Chiral Anomaly

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Outline of the Talk

Introduction: What do they mean by „anomaly“?

Main Part:

- QM formulation of current symmetries
- Shifting integration variable
- Quantum fluctuation violates axial vector current
- Consequences of the anomaly

Summary

Why „anomaly“?

Classical symmetry

QM Symmetry

Transformation

$$\varphi \to \varphi + \delta\varphi$$

Action

$$S(\varphi)$$

invariant

Path Integral

$$\int D\varphi \cdot e^{iS(\varphi)}$$

invariant

- The idea classical symmetry ⟶ QM symmetry

became a comfortable „habit“ just think of rotations,translations etc.

- Not necessarily! QM fluctuations cause „anomalies“ = unexpected truth

Classical treatment

- Consider a Lagrangian for a single massless fermion

$$\mathcal{L} = \overline{\psi} i \gamma^\mu \partial_\mu \psi$$

<u>Transformations</u>

$$\psi \to e^{i\Theta}\psi$$

$$\psi \to e^{i\Theta\gamma^5}\psi$$

<u>Conserved Quantites:</u>

Vector Current $\quad J^\mu = \overline{\psi}\gamma^\mu\psi \quad \partial_\mu J^\mu = 0$

Axial-Vector Current $\quad J_5^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi \quad \partial_\mu J_5^\mu = 0$

QM formulation

- Consider two fermion-antifermion pairs being created at x_1 and x_2 respectively

- The corresponding vector currents are

$$J^\mu(x_1) = \overline{\psi}(x_1)\gamma^\mu\psi(x_1) \qquad J^\nu(x_2) = \overline{\psi}(x_2)\gamma^\nu\psi(x_2)$$

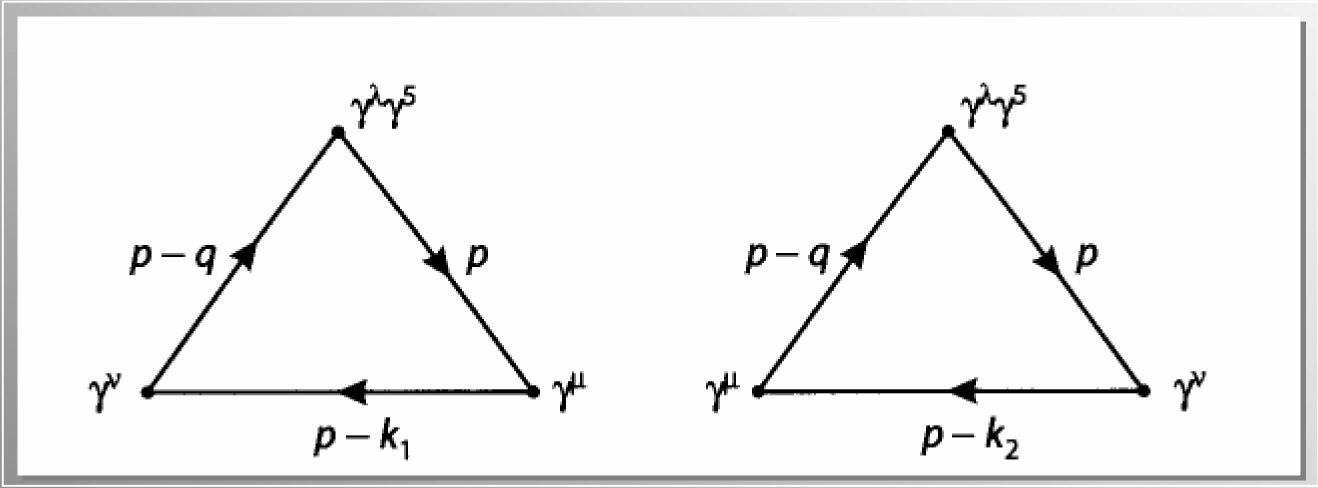
- The axial vector current $J^\lambda(0) = \overline{\psi}(0)\gamma^\lambda\gamma^5\psi(0)$, $x_3 = 0$

- The amplitude for this process is described by

$$\left\langle 0 \left| T J_5^\lambda(0) J^\mu(x_1) J^\nu(x_2) \right| 0 \right\rangle$$

QM formulation

- $\langle 0 \left| T J_5^\lambda(0) J^\mu(x_1) J^\nu(x_2) \right| 0 \rangle$ evaluation of the amplitude results in triangle diagrams



- Applying Feynmanrules for Fermions and taking the bose statistics for possible vector bosons into account, we get the amplitude in momentum space

QM formulation

$$\Delta^{\lambda\mu\nu}(k_1, k_2) = (-1)i^3 \int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k_1}} \gamma^\mu \frac{1}{\not{p}}$$

$q = k_1 + k_2$

$$+\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\mu \frac{1}{\not{p} - \not{k_2}} \gamma^\nu \frac{1}{\not{p}})$$

- Classical the vector currents and axial-vector current should be conserved simultaneously

- Since $\Delta^{\lambda\mu\nu}$ is written in momentum space and the currents can be Fouriertransformed

$$J^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} J^\mu(k)$$

QM formulation

The current conservation $\partial_\mu J^\mu(x_1) = 0$ $\partial_\nu J^\nu(x_2) = 0$

$$\partial_\lambda J^{5\lambda}(0) = 0$$

Is aquivalent to $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$ $k_{2\nu}\Delta^{\lambda\mu\nu} = 0$

$$q_\lambda \Delta^{\lambda\mu\nu} = 0$$

Importance of current conservation

Vector

- $Q = \int d^3x J^0$ Counts the fermion number

- Introducing of a photon should cause no troubles:

$$\frac{i}{k_1^2}[(\zeta \frac{k_{1\mu}k_{1\rho}}{k_1^2} - g_{\mu\rho})]$$

$$k_{1\mu}\Delta^{\lambda\mu\nu} = 0$$

$$-\frac{ig_{\mu\rho}}{k_1^2}$$

Axial vector

- Real world fermions are not massless

For $m_f \neq 0$ even the classical symmetry is not valid

It‘s getting serious…

- Naive way to evaluate the integral

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = (-1)i^3 \int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{\not k_1}{\not p - \not k_1}\frac{1}{\not p}$$

$$+\gamma^\lambda\gamma^5 \frac{\not k_1}{\not p - \not q}\frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{\not p})$$

Substitute

1.Term: $\not k_1 \to \not p - (\not p - \not k_1)$

2.Term: $\not k_1 \to (\not p - \not k_2) - (\not p - \not q)$

$$q = k_1 + k_2$$

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = i \int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1} - \gamma^\lambda\gamma^5 \frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{\not p})$$

↑

$$\frac{1}{(\not p - \not k_2) - \not k_1}$$

Variable shifting

- If we could shift the integration variable in the 2. term

$\not{p} \to \not{p} - \not{k}_1$ then we would have $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$

- But is the shift of the integral variable allowed here?

- Consider some arbitrary function $f(p)$ then the difference is

$$\int_{-\infty}^{+\infty} dp(f(p+a) - f(p)) = \int_{-\infty}^{+\infty} dp(a\frac{d}{dp}f(p) + \ldots)$$

$$= a(f(+\infty) - f(-\infty)) + \ldots$$

Variable shifting

- The integral have to be convergent or logarithmically divergent. Here we deal with an lineary divergent integral!

- Rotate the Feynman integrand to d-dimensional Euclidian space:

$$\int d^d_E p[f(p+a) - f(p)] = \int d^d_E p[a^\mu \partial_\mu f(p) + \ldots]$$

applying Gauss‘s theorem we get

$$\int d^d_E p[f(p+a) - f(p)] = \lim_{P\to\infty} a^\mu (\tfrac{P_\mu}{P}) f(p) S_{d-1}(P)$$

Now rotating back and addopting for a 4-dim Minkowski space

$$\int d^4 p[f(p+a) - f(p)] = \lim_{P\to\infty} i a^\mu (\tfrac{P_\mu}{P}) f(p)(2\pi^2 P^3)$$

…almost hopeless

- Obviously is $f(p) = tr(\gamma^\lambda \gamma^5 \frac{1}{\not p - \not k_2} \gamma^\nu \frac{1}{\not p})$

applying trace theorem $= \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} p_\sigma}{(p-k_2)^2 p^2}$

with $a^\mu = -k_1^\mu$ and plugging $f(p)$ into the integral

$$\longrightarrow \quad \boxed{k_{1\mu}\Delta^{\lambda\mu\nu} = \frac{i}{8\pi^2}\epsilon^{\lambda\tau\sigma} k_{1\tau} k_{2\sigma} = \partial_\mu J^\mu \neq 0}$$

Don‘t forget the physics behind!

Use freedom of choice in labeling internal momenta $\not{p} \to \not{p} + \not{a}$

Calculate $\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2)$

Now $\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2}\epsilon^{\sigma\nu\mu\lambda}a_\sigma + \{\mu, k_1 \leftrightarrow \nu, k_2\}$

k_1, k_2 are independent $a = \alpha(k_1 + k_2) + \beta(k_1 - k_2)$

$$\boxed{\Delta^{\lambda\mu\nu}(a, k_1, k_2) = \Delta^{\lambda\mu\nu}(k_1, k_2) + \frac{i\beta}{4\pi^2}\epsilon^{\lambda\mu\nu\sigma}(k_1 - k_2)_\sigma}$$

Fixing the paramter

At the same time demand a physical reasonable outcome by fixing the parameter β

$$k_{1\mu}\Delta^{\lambda\mu\nu} = 0$$

and recalling that

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2}\epsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

if we choose

$$\beta = -\frac{1}{2}$$

we get vector current conservation

$$k_{1\mu}\Delta^{\lambda\mu}(a, k_1, k_2) = 0$$

Breaking with „bad habits“

After insisting on vector current conservation check if $q_\lambda \Delta^{\lambda\mu\nu} = 0$

$$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) + \frac{i}{4\pi^2}\epsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}$$

Following the same calculation pattern as above we get

$$\boxed{q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{4\pi^2}\epsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma}}$$

Fixing the paramter β once we prevent the conservation of the axial current

$$q_\lambda \Delta^{\lambda\mu\nu} \neq 0$$

In QM the vector current and the axial vector current can‘t be conserved at the same time

$\longrightarrow$ Choose physical correct option!

Consequences

1. Gauged theory: $\mathcal{L} = \overline{\psi} i \gamma^{\mu} (\partial_{\mu} - ieA_{\mu})\psi$

with A_{μ} the photon field

The result can be written then

Classical $\partial_{\mu} J_5^{\mu} = 0$

QM $\partial_{\mu} J_5^{\mu} = \frac{e^2}{(4\pi)^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$ $\longrightarrow$ Operator capable of producing two photons

2. Experimentally found $\pi^0 \rightarrow 2\gamma$ decay can not occure according to classical view

QM resolves the apparent contradiction and leads to the correct result

Consequences

3. Correction to the classical result in the presence of a mass term:

The symmetry for $\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ with $\psi \to e^{i\Theta\gamma^5}\psi$

is spoiled by the mass term

Classically $\partial_\mu J_5^\mu = 2m\overline{\psi}i\gamma^5\psi$

In gauged theory with not vanishing mass the qm fluctuation give rise to an additional correction term

$$\partial_\mu J_5^\mu = 2m\overline{\psi}i\gamma^5\psi + \frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

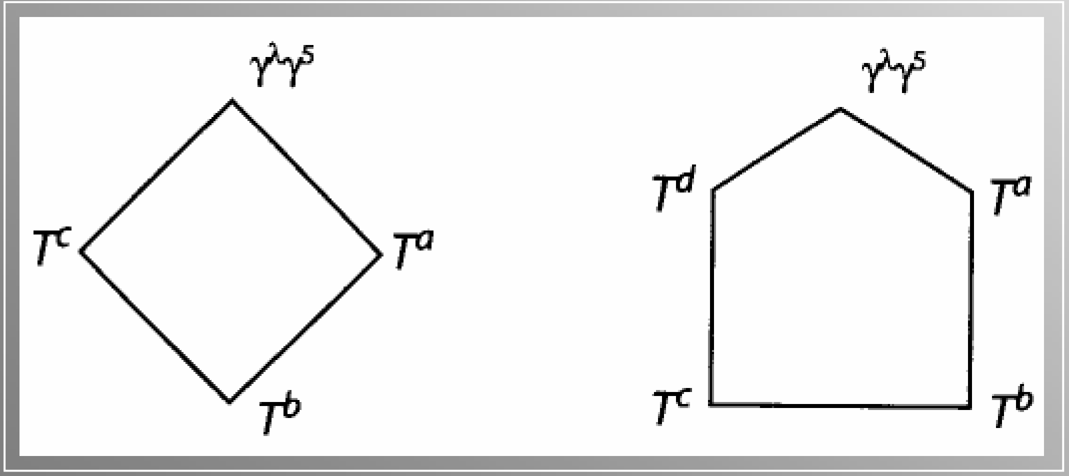
Consequences

4. Square and pentagon anomaly:

Occure in non-abelian gauge theories $\mathcal{L} = \overline{\psi} i\gamma^\mu(\partial_\mu - ieA_\mu^a T^a)\psi$

$$\partial_\mu J_5^\mu = \frac{g^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma} tr F_{\mu\nu}F_{\lambda\sigma} \quad \text{with} \quad F_{\mu\nu} = F_{\mu\nu}^a T^a$$

contains terms cubic and quartic in A

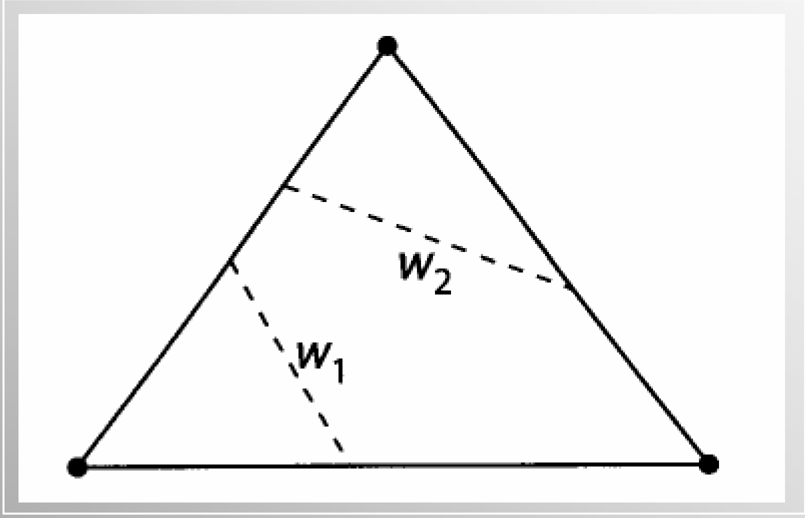


Consequences

5. Nonrenormalization

Add a scalar field to the free Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi + f\varphi\overline{\psi}\psi \longrightarrow$$ higher order loop diagrams



multiply $q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2)$ with

$1 + h(f, e, \ldots)$

Adler and Barden: $h(f, e \ldots) = 0$

Nonrenormalization

Heuristically: seven fermion propagators

$\longrightarrow$ Integral sufficiantly convergent

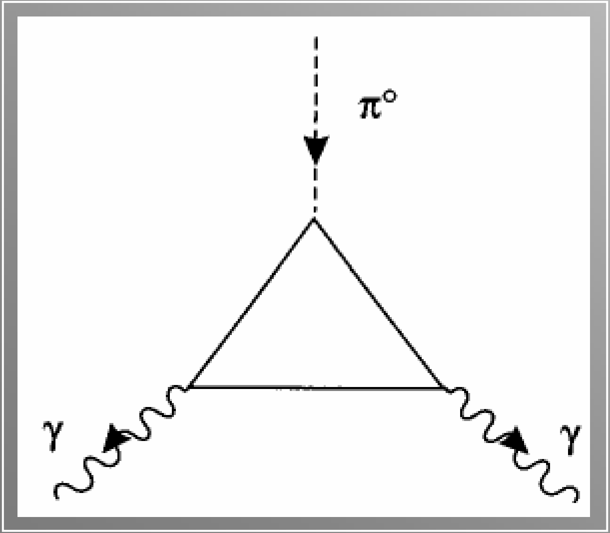
$\longrightarrow$ Shift of integration variable allowed $\longrightarrow$ $q_\lambda \Delta^{\lambda\mu\nu}_{3loops}(k_1, k_2; W_1, W_2) = 0$

Consequences

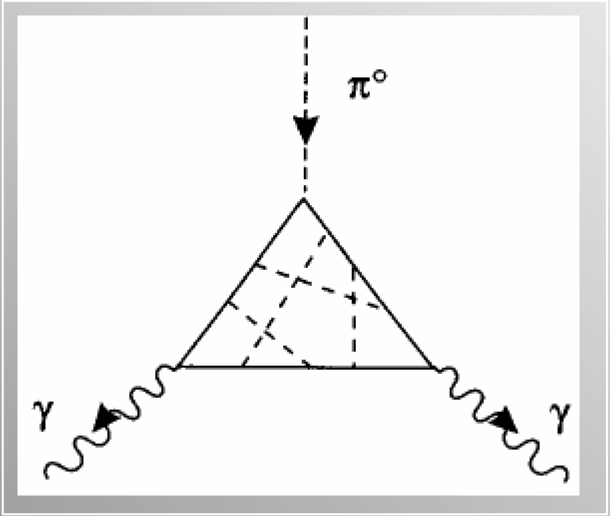
6. Suggestion about quarks:

Nonrenormalization → decay amplitude for $\pi^0 \to 2\gamma$

the only contributing



Quark model of 60's with an infinite number of Feynman diagrams



Summary

- Classical symmetry does not imply the same symmetry in quantum mechanics

- The shift of the integration variable is not allowed in (lineary) divergent integrals

- In Qm the vector current is conserved and the axial vector current is not conserved

- QM confirmation of the neutral pion decay

Thank you for your attention

Appendix A

- Polarization and Photon propogator:

propagator for massless spin 1 bosons: $\frac{i}{k_1^2}[\zeta(\frac{k_{1\mu}k_{1\rho}}{k_1^2} - g_{\mu\rho})]$

Polarization vectors $\epsilon_\mu(p), \epsilon_\mu^\star(p)$

Relation between propagator and polarization vectors $\sum \epsilon_\mu(p)\epsilon_v^\star(p) = g_{\mu\nu}$

Appendix B

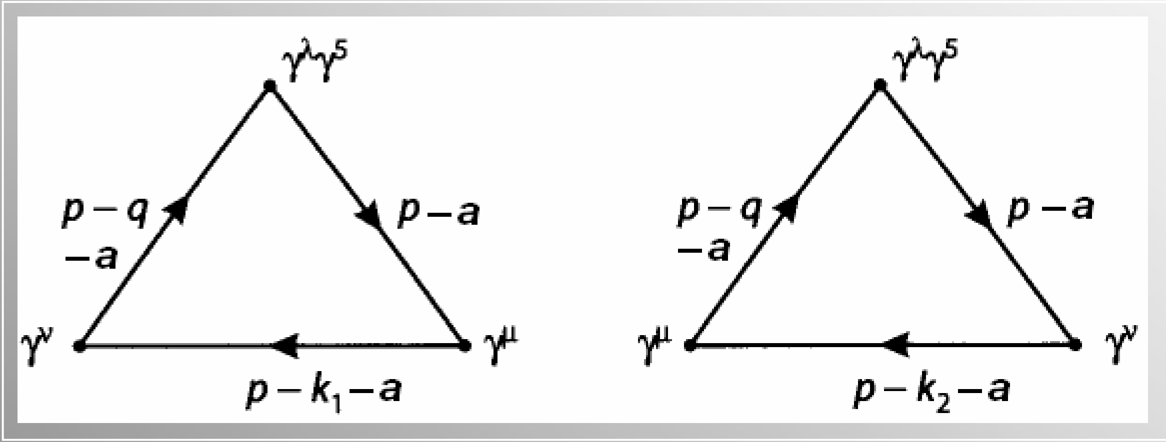
Gauß Integral:

$$\int d_E^d p a^\mu \partial_\mu f(p) = \int d(S_{d-1})_\mu a^\mu f(p) =$$

$$= \int d(S_{d-1}) \frac{p_\mu}{p} a^\mu f(p) = \lim_{P\to\infty} a^\mu (\frac{P_\mu}{P}) f(p) S_{d-1}(P)$$

Appendix C

- Shifting internal momenta:



$$\Delta^{\lambda\mu\nu}(a,k_1,k_2) = (-1)i^3\int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda\gamma^5 \frac{1}{\not p+\not a-\not q}\gamma^\nu \frac{1}{\not p+\not a-\not k_1}\gamma^\mu \frac{1}{\not p+\not a})$$

$$+\{\mu, k_1 \leftrightarrow \nu, k_2\}$$