Effective Potential

Seminar Talk by Kilian Rosbach

Seminar Series on
Theoretical Particle Physics

Prof. M. Drees and Prof. H.-P. Nilles

June 22nd, 2006

Talk outline

- Motivation
- Definition of the effective potential
- Application to ϕ^4-theory
  - Loop correction to the classical potential
  - Renormalization
- Interpretation
- (0+1) dimensional spacetime
- Summary

Motivation (1/4)

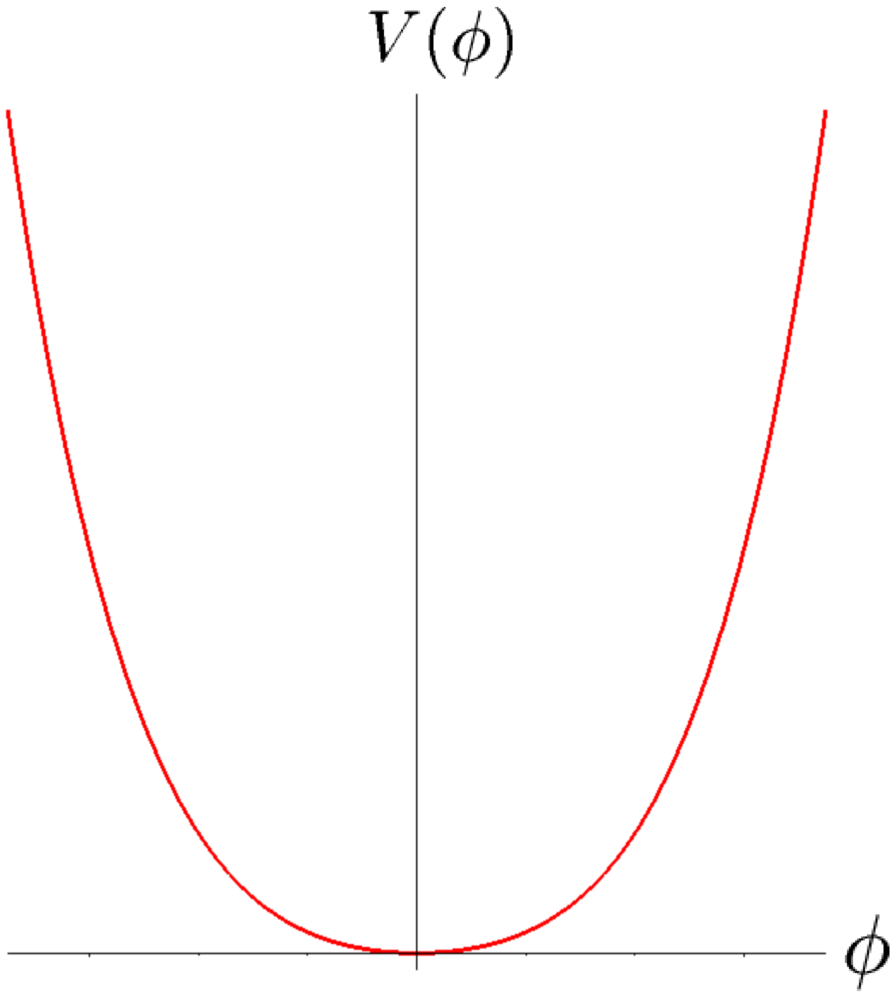
- A symmetry can be spontaneously broken when there is no unique ground state.
- Lagrangian-density in ϕ^4-theory (scalar particle):

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

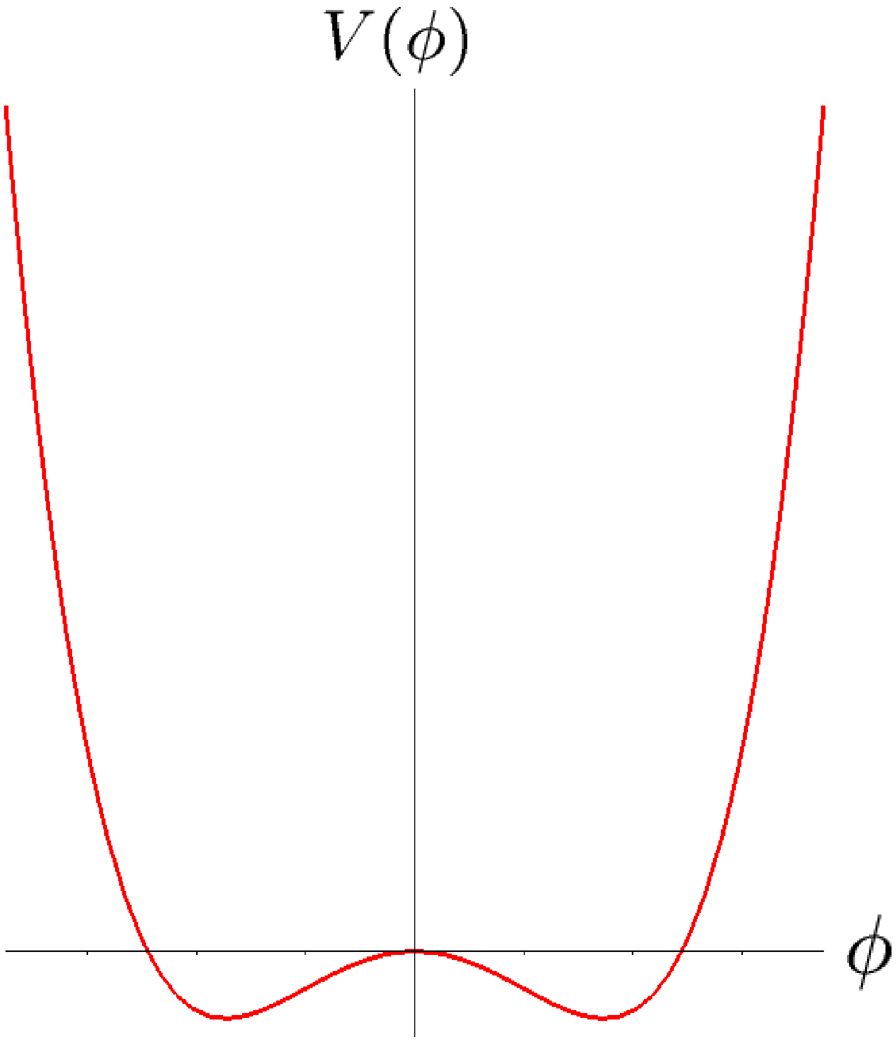
- We take a closer look at $V(\phi)$:
  - For $\mu^2>0$: (One) minimum: $V'(\phi_{min})=0 \leftrightarrow \phi_{min}=0$
  - For $\mu^2<0$: Broken symmetry (Mexican hat)
- In this context V is called **classical potential**.

Motivation (2/4)

Classical potential for $\mu^2>0$



... and $\mu^2<0$

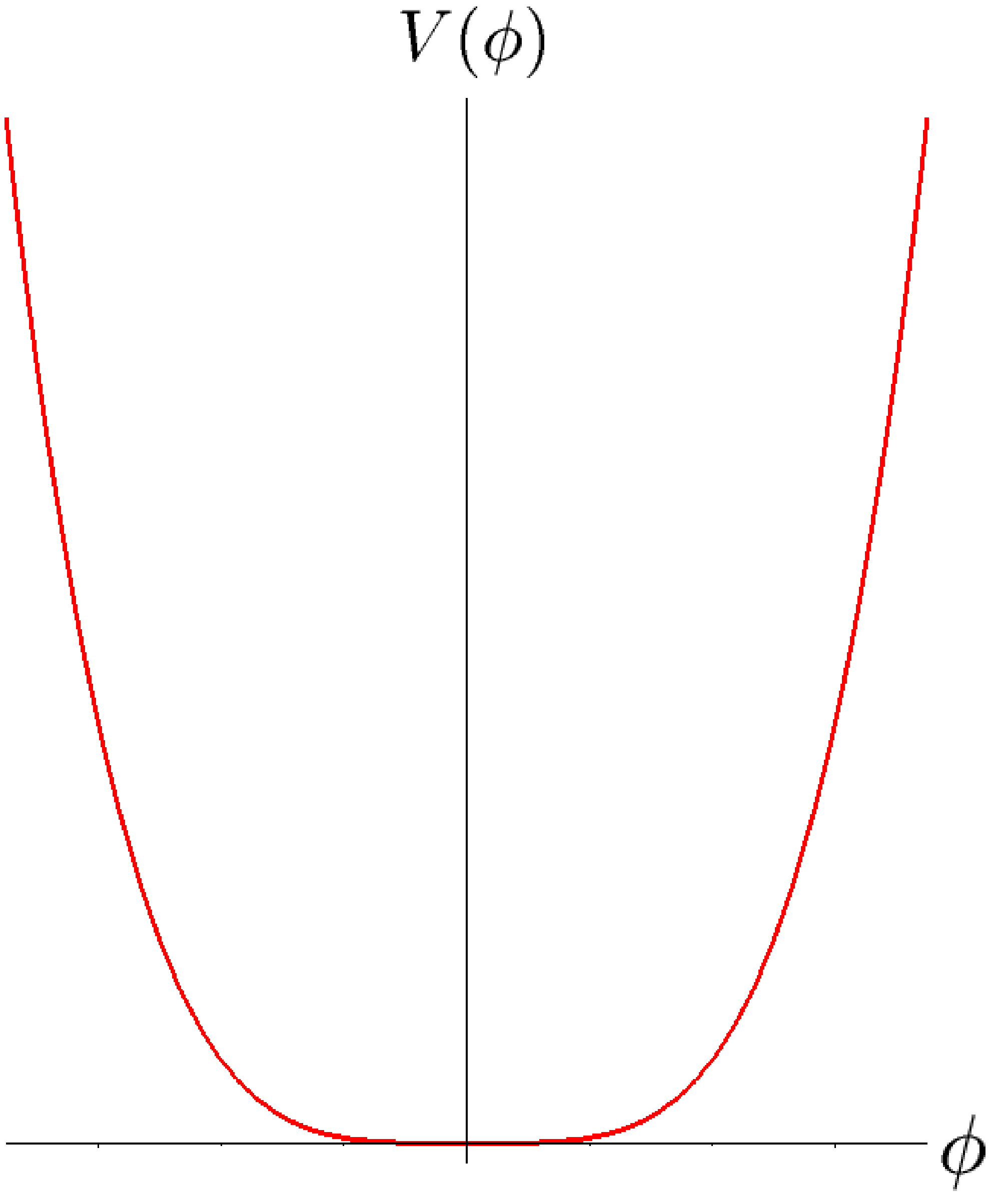


Motivation (3/4)

- Question:
  What happens at $\mu^2=0$?

- Or, more specifically:

  Is the classical interpretation of the potential still valid?



Motivation (4/4)

- We are "just on the edge of symmetry breaking" !
- Classically: $V(\phi) = \lambda/4!\ \phi^4$
  Still one ground state → Symmetry unbroken
- As we shall see:
  Quantum fluctuations "push the system over the edge"
- To prove this, we will replace the classical potential by an **effective potential**, which contains corrections of $\mathcal{O}(\hbar)$

Definition of the effective potential (1/5)

- Starting point is the vacuum-to-vacuum-amplitude

$$Z = e^{iW[J]} = \int \mathcal{D}\phi\, e^{i\int \mathcal{L}[\phi] + J(x)\phi(x)\, d^4x}$$

- J describes sources and sinks, where particles are created and annihilated – it is only a tool!
- The generating functional W[J] can be expanded in J.

The first order term

$$\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z(J=0)} \int \mathcal{D}\phi\, e^{i\int \mathcal{L}[\phi] + J(x)\phi(x)\, d^4x}\, \phi(x)$$

describes the expectation value of ϕ: $\langle 0|\hat{\phi}|0\rangle_J =: \phi_v(x)$

Definition of the effective potential (2/5)

$$\phi_v(x) \equiv \frac{\delta W[J]}{\delta J(x)}$$

- $\phi_v(x)$ is the expectation value of a field-operator. For $J \to 0$ it is a vacuum expectation value !
  - We will later demand that it does not depend on x.
- Also consider the Legendre transform of J:

$$\Gamma[\phi_v] = W[J] - \int d^4x\, J(x)\phi_v(x)$$

- As it should be, Γ does not depend on J:

$$\frac{\delta\Gamma[\phi_v]}{\delta J(y)} = 0$$

Definition of the effective potential (3/5)

- We calculate the functional derivative w.r.t. ϕ_v and find

$$J(y) = -\frac{\delta \Gamma[\phi_v]}{\delta \phi_v(y)}$$

so that we now have the "dual" relations

$$\phi_v(x) = \frac{\delta W[J]}{\delta J(x)} \quad \leftrightarrow \quad J(y) = -\frac{\delta \Gamma[\phi_v]}{\delta \phi_v(y)}$$

Definition of the effective potential (4/5)

- Ansatz for Γ:

$$\Gamma[\phi_v] = \int d^4x \left[-\mathcal{E}(\phi_v) + X(\phi_v)(\partial\phi_v)^2 + Y(\phi_v)(\partial\phi_v)^4 + (...)\right]$$

- Assume that vev ϕ_v is constant (which is sensible – otherwise it would not be translationally invariant)
- Then:

$$\Gamma[\phi_v] = \int d^4x \left[-\mathcal{E}(\phi_v)\right]$$

$$J(y) = -\frac{\delta\Gamma[\phi_v]}{\delta\phi_v(y)} = \mathcal{E}'(\phi_v)$$

Definition of the effective potential (5/5)

- Remembering J was only a tool, we let $J \to 0$
  Without external sources (or sinks) we find:

$$\mathcal{E}'(\phi_v) = 0$$

- Compare this with the classical theory! There, the ground state was determined by $V'(\phi_{min})=0$
- $\mathcal{E}$ should be interpreted as a potential
  - We call it V_{eff} – the **effective potential**
  - We will now investigate further to find that

$$V_{eff}(\phi) = V(\phi) + \mathcal{O}(\hbar)$$

Loop correction to the classical potential (1/3)

- To get any result, we have to evaluate W[J].
- Use the "steepest descent" approximation:
  - Insert $\hbar$ back into Z:

$$Z = e^{\frac{i}{\hbar} W[J]} = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} \int \mathcal{L}[\phi] + J(x)\phi(x) d^4x}$$

  - $\hbar$ is small → exponential dominated by small values of S+Jϕ
  - Expand ϕ around minimum: $\phi = \phi_s + \tilde{\phi}$
  - The minimum of S+Jϕ can be found using the Euler-Lagrange equations with an additional term for J:

$$\partial^2 \phi_s + V'[\phi_s(x)] = J(x)$$

Loop correction to the classical potential (2/3)

- The result of the approximation is:

$$W[J] = S[\phi_s] + \int d^4x \, [J(x)\phi_s(x)] + \frac{i\hbar}{2} Tr \log(\partial^2 + V''[\phi_s]) + \mathcal{O}(\hbar^2)$$

- To leading order: $\phi_v = \phi_s$
- To evaluate the log, we again assume $\phi_v = \text{const.}$

$$Tr \log(\partial^2 + V''[\phi]) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \log\left[-k^2 + V''(\phi)\right]$$

Loop correction to the classical potential (3/3)

- Putting it all together, we find the

COLEMAN-WEINBERG effective potential

$$V_{eff}(\phi_v) = V(\phi_v) - \frac{i\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \log\left[\frac{k^2 - V''(\phi_v)}{k^2}\right] + \mathcal{O}(\hbar^2)$$

- Describes $\mathcal{O}(\hbar)$ correction to classical potential.
- We have added a constant to make log dimensionless.

Renormalization (1/4)

- (From now on: write ϕ instead of ϕ_v)
- The Coleman-Weinberg effective potential has one problem: The integral is quadratically divergent!
- We will renormalize the bare quantities by introducing counterterms in the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - A(\partial\phi)^2 - B\phi^2 - C\phi^4$$

- These terms then also appear in the effective potential:

$$V_{eff}(\phi) = V(\phi) - \frac{i\hbar}{2}\int \frac{d^4k}{(2\pi)^4} \log\left[\frac{k^2 - V''(\phi)}{k^2}\right] + A(\partial\phi)^2 + B\phi^2 + C\phi^4 + \mathcal{O}(\hbar^2)$$

Renormalization (2/4)

- We introduce a cut-off Λ and integrate up to $k^2=\Lambda^2$
- Result:

$$V_{eff}(\phi) = V(\phi) + \frac{\Lambda^2}{32\pi^2} V''(\phi) - \frac{[V''(\phi)^2]}{64\pi^2} \log \frac{e^{\frac{1}{2}}\Lambda^2}{V''(\phi)} + B\phi^2 + C\phi^4$$

- Since ϕ=const, the counterterm $A(\partial\phi)^2$ is not needed.
- Now we can "watch renormalisation theory at work":
  - All cut-off dependent terms are of order ϕ^2 or ϕ^4 and can be absorbed into B and C!

Renormalization (3/4)

- Back to our introductory question: What happens at μ=0?

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \quad \Rightarrow \quad V''(\phi) = \frac{\lambda}{2}\phi^2$$

$$V_{eff}(\phi) = \left(\frac{\Lambda^2}{64\pi^2}\lambda + B\right)\phi^2 + \left(\frac{\lambda}{4!} + \frac{\lambda^2}{(16\pi)^2}\log\frac{\phi^2}{\Lambda^2} + C\right)\phi^4 + \mathcal{O}(\lambda^3)$$

- To determine B and C we have to find **2 renormalization conditions**.

- The term ~ϕ^2 is the (renormalized) mass – we want the mass to stay zero, so we impose the first condition:

$$\left.\frac{d^2V_{eff}}{d\phi^2}\right|_{\phi=0} = 0$$

Renormalization (4/4)

- We cannot do the same for ϕ^4 because the derivative depends on log ϕ, which is undefined at $\phi=0$.
- The second condition has to be fixed by the coupling λ(M) at some mass scale M:

$$\left.\frac{d^4 V_{eff}}{d\phi^4}\right|_{\phi=M} = \lambda(M)$$

- Doing some algebra we arrive at our final result:

$$V_{eff}(\phi) = \frac{\lambda(M)}{4!}\phi^4 + \frac{\lambda(M)^2}{(16\pi)^2}\phi^4\left(\log\frac{\phi^2}{M^2} - \frac{25}{6}\right) + \mathcal{O}\left[\lambda(M)^3\right]$$

  - This depends on physical quantities only! No C, no Λ !

Interpretation (1/3)

$$V_{eff}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2}{(16\pi)^2}\phi^4\left(\log\frac{\phi^2}{M^2} - \frac{25}{6}\right) + \mathcal{O}\left[\lambda^3\right]$$

Comparing with the classical potential

$$V(\phi) = \frac{\lambda}{4!}\phi^4$$

we see that the correction to the potential is of the form

$$\lambda^2\phi^4\log\frac{\phi}{M}$$

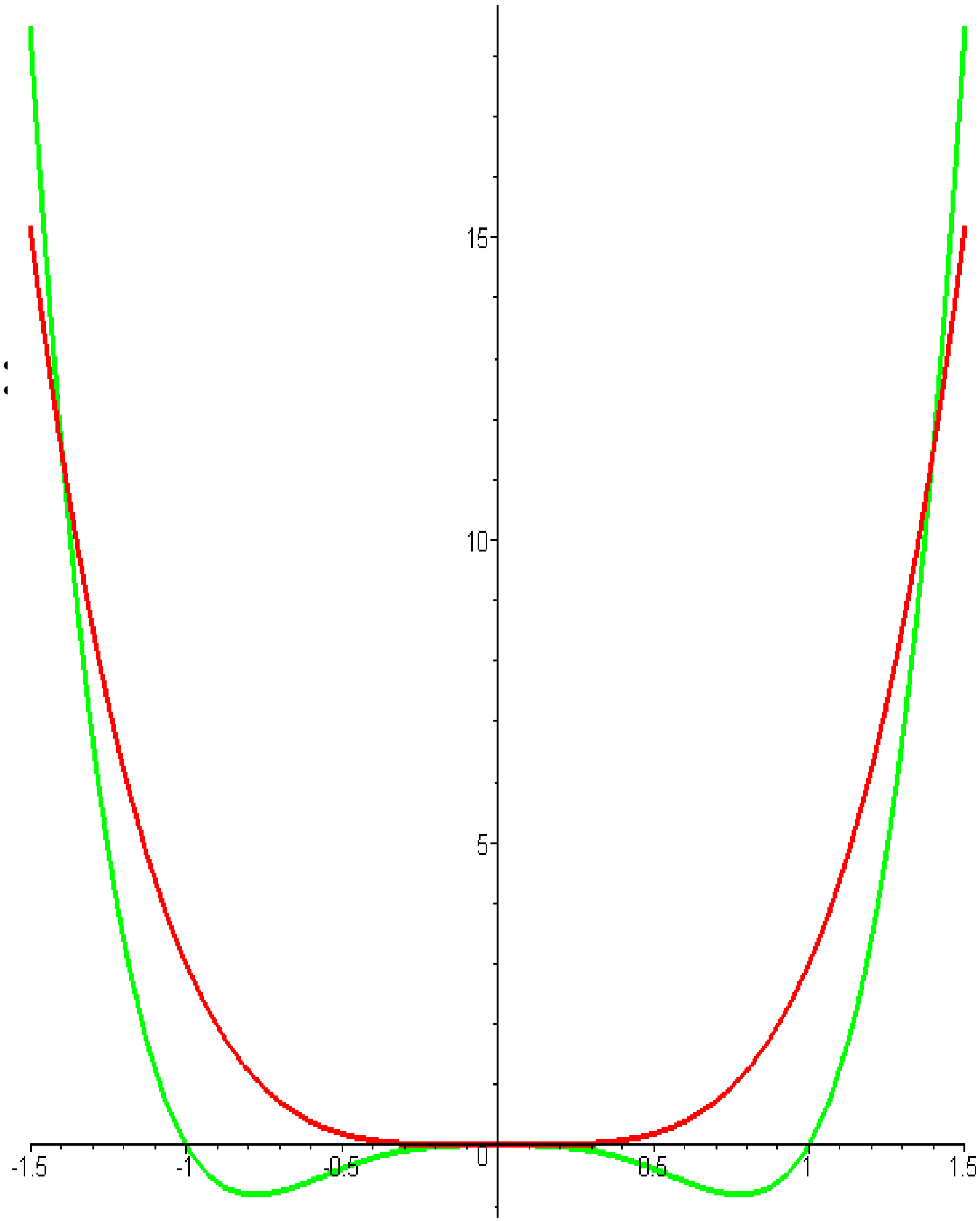
$$\Rightarrow \quad \frac{\mathcal{O}(\hbar^1)}{\mathcal{O}(\hbar^0)} \quad \propto \quad \lambda\log\frac{\phi}{M}$$

Interpretation (2/3)

- We plot both potentials – and find the answer to our introductory question:
- At μ=0, the symmetry is broken!

Conclusion:

Quantum fluctuations can generate spontaneous symmetry-breakdown!



Interpretation (3/3)

- One word of warning:
  The position of the minima determined in this way cannot be taken too seriously because they are in a range where the expansion parameter $\lambda \log(\phi/M)$ is of order unity.
- There are ways around this problem, but these are not part of this talk.

Application in (0+1)-dimensional spacetime (1/2)

Reminder: (0+1)-dim. spacetime is just Quantum Mechanics!

- The Coleman-Weinberg effective potential now reads:

$$V_{eff}(\phi) = V(\phi) + \frac{\hbar}{2} \int \frac{dk}{2\pi} \log\left(\frac{k^2 + V''(\phi)}{k^2}\right) + \mathcal{O}(\hbar^2)$$

- The integral is convergent and we obtain:

$$V_{eff}(\phi) = V(\phi) + \frac{\hbar}{2}\sqrt{V''(\phi)} + \mathcal{O}(\hbar^2)$$

Application in (0+1)-dimensional spacetime (2/2)

- Apply this result to an Harmonic Oscillator:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\omega^2\phi^2$$

$$\Rightarrow \; V''(\phi) = \omega^2$$

$$V_{eff}(\phi) = \frac{1}{2}\omega^2\phi^2 + \frac{\hbar}{2}\omega + \mathcal{O}(\hbar^2)$$

- For the ground state we find the familiar result:

$$V_{eff}(\phi = 0) = \frac{\hbar}{2}\omega$$

Summary

- We developed the effective potential formalism to calculate radiative corrections to the classical potential.
- For the ϕ^4-theory of a massless scalar we discovered a radiatively induced symmetry breakdown.
- In our example, the results of the effective potential formalism are consistent with quantum mechanics.

References

- A. Zee – Quantum Field Theory in a Nutshell
- L.H. Ryder – Quantum Field Theory
- S. Coleman – Aspects of Symmetry
- T. Cheng, L. Li – Gauge Theory of Elementary Part.Ph.