The Optical Theorem and Partial Wave Unitarity

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Seminar on Theoretical Particle Physics
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June 29th 2006

What is the **Optical Theorem**?

Motivation:

The Optical Theorem:

$$\text{Im } f(\theta = 0) = \frac{|\mathbf{k}|}{4\pi}\sigma_{\text{tot}} \tag{1}$$

Is there a more general concept behind this?

$\Rightarrow$ Answer: **Yes!**

short: The optical theorem follows directly from the unitarity of the S-matrix.

$\Rightarrow$ How can we define the S-matrix in a physical meaningful way?

Let's take a look at this...

Definition of one-and many-Particle-States

One-particle-state:

$$|\phi\rangle(t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \cdot \frac{\phi(\mathbf{k})}{\sqrt{2E(k)}} |\mathbf{k}\rangle(t) \tag{2}$$

Multi-particle-state:

$$|\phi_1 \phi_2 \ldots\rangle(t) \equiv |\{\phi_f\}\rangle(t) = \prod_f \int \frac{\mathrm{d}^3 k_f}{(2\pi)^3} \cdot \frac{\phi_f(\mathbf{k}_f)}{\sqrt{2E_f}} |\{\mathbf{k}_f\}\rangle(t) \tag{3}$$

with $|\{\mathbf{k}_f\}\rangle \equiv |\mathbf{k}_1 \mathbf{k}_2 \ldots\rangle$

Definition of one-and many-Particle-States

One-particle-state:

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Multi-particle-state:

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with $|\{\mathbf{k}_f\}\rangle \equiv |\mathbf{k}_1\mathbf{k}_2...\rangle$

The starting point

Question:

What is the transition probability for the scattering of two particles A and B into a many-particle-state $|\{\phi_f(t)\}\rangle$?

transition propability

$$\Rightarrow \quad \mathcal{P}(t_2, t_1) = \Big| \underbrace{\langle \{\phi_f\}(t_2)}_{\text{out}} \Big| \underbrace{\phi_A \phi_B(t_1)\rangle}_{\text{in}} \Big|^2 \tag{4}$$

$\Rightarrow$ We have to compute $\langle \{\mathbf{p}_f\}(t_2) | \mathbf{k}_A \mathbf{k}_B(t_1)\rangle$

Definition of the S-matrix

The S-matrix relates the **in**coming particles (coming from $t_1 \rightarrow -\infty$) and the **out**going particles (going to $t_2 \rightarrow +\infty$).

Definition: The S-matrix:

$$\langle\{\mathbf{p}_f\}|S|\mathbf{k}_A\mathbf{k}_B\rangle \equiv \lim_{t\rightarrow+\infty}\langle\{\mathbf{p}_f\}(t)|\mathbf{k}_A\mathbf{k}_B(-t)\rangle \tag{5}$$

Some properties of the S-matrix:

- S have to be unitary ($S^\dagger S = 1$)
- S is the identity if the particles do not interact between each other

Definition of the T-matrix

Definition: The T-matrix:

$$\boxed{S \equiv 1 + iT} \quad (6)$$

So, T is the interesting part of the interaction process ($\Rightarrow$ shows if something interacts).

Some properties of the T-matrix:

- S unitary $\Leftrightarrow$ $S^\dagger S = 1 \Rightarrow T^\dagger T = -i(T - T^\dagger)$
- $T = 0$ if the particles do not interact between each other

Define the invariant matrix-element $\mathcal{M}(\mathbf{k}_A \mathbf{k}_B \rightarrow \{\mathbf{p}_f\})$ by

$$\langle \{\mathbf{p}_f\} | iT | \mathbf{k}_A \mathbf{k}_B \rangle = i\mathcal{M}(\mathbf{k}_A \mathbf{k}_B \rightarrow \{\mathbf{p}_f\}) \, (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f) \quad (7)$$

$\mathcal{M}$ is proportional to the scattering amplitude f.

How does the Cross-Section σ depends on $\mathcal{M}$?

Transition propability

At first we decide the probability that the initial state $|\phi_A \phi_B\rangle$ becomes scattered into the final momentum-state $|\{\mathbf{p}_f\}\rangle$ (that means in a small region $\prod_f \mathrm{d}^3 p_f$).
Therefore:

$$\mathcal{P}(AB \to \{\mathbf{p}_f\}, b) = \prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} |\langle \{p_f\} | \phi_A \phi_B \rangle (b)|^2 \qquad (8)$$

with b as impact-parameter. Definition of the cross-section

$$\sigma = \frac{N_{\mathrm{sc}}}{n_B N_A} = \int \mathrm{d}^2 b \mathcal{P}(b) \qquad (9)$$

($N_{sc} \triangleq \#$ scattered particles, $n_B \triangleq$ number density, $N_A \triangleq \#$ incoming particles)

How does the Cross-Section σ depends on $\mathcal{M}$?

Total Cross-Section

$$\sigma_{\text{tot}} = \left(\prod_f \int \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^{(4)}(P - \sum_f p_f)\right) \times \frac{|\mathcal{M}(\mathbf{p}_A \mathbf{p}_B \to \{\mathbf{p}_f\})|^2}{2E_A 2E_B |v_A - v_B|}$$

with $P = p_A + p_B$ and $v_i = k_i^z / E_i, \; i = A, B$. Now, we can follow the optical theorem quite easy...

The Optical Theorem

The Optical Theorem:
We know $S^\dagger S = 1 \;\Rightarrow\; T^\dagger T = -i(T - T^\dagger)$ an so we can calculate the scattering amplitude for the process $k_1 k_2 \to p_1 p_2$.

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \left(\prod_{f=1}^{n} \int \frac{\mathrm{d}^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \langle p_1 p_2 | T^\dagger | \{q_f\} \rangle \langle \{q_f\} | T | k_1 k_2 \rangle \tag{10}$$

This gives us

$$\begin{aligned} &-i(\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)) \\ &= \sum_n \left(\prod_{f=1}^{n} \int \frac{\mathrm{d}^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \mathcal{M}(p_1 p_2 \to \{q_f\}) \mathcal{M}^*(k_1 k_2 \to \{q_f\}) \\ &\qquad \cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f) \end{aligned} \tag{11}$$

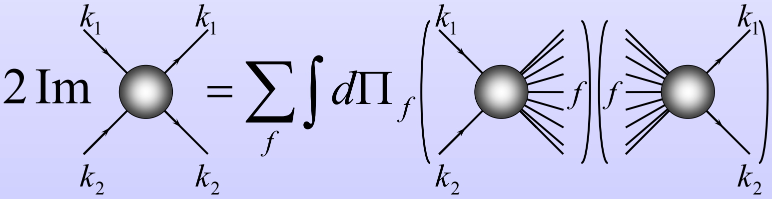
The Optical Theorem

The optical theorem relates the forward scattering amplitude to the cross-section.

$\Rightarrow \quad k_i = p_i, i = 1, 2$ for forward scattering

$$2\mathrm{Im}\ \mathcal{M}(k_1 k_2 \to k_1 k_2) \quad = \quad \sum_n \underbrace{\left(\prod_{f=1}^{n} \int \frac{\mathrm{d}^3 q_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f)\right)}_{\equiv \int \mathrm{d}\Pi_n}$$

$$\times \left|\mathcal{M}(k_1 k_2 \to \{q_f\})\right|^2$$



The Optical Theorem

Put in the relation for the total cross-section

$$\Rightarrow 2\mathrm{Im}\ \mathcal{M}(\mathrm{k_1 k_2} \rightarrow \mathrm{k_1 k_2}) = 2\mathrm{E_A} 2\mathrm{E_B} |\mathrm{v_A} - \mathrm{v_B}| \sigma_{\mathrm{tot}} \tag{12}$$

go into the CM-system
$(\mathbf{p}_A + \mathbf{p}_B = 0 \;\Rightarrow E_{\mathrm{CM}} = E_A + E_B,\; \mathbf{P}_{CM} = \mathbf{p}_A = -\mathbf{p}_B)$

Optical Theorem (Standardform)

$$\mathrm{Im}\ \mathcal{M}(\mathrm{k_1 k_2} \rightarrow \mathrm{k_1 k_2}) = 2\mathrm{E_{CM} P_{CM}} \sigma_{\mathrm{tot}} \tag{13}$$

The Optical Theorem for Feynman Diagrams

The Optical Theorem for Feynman Diagrams

We can derive $\mathcal{M}$ by the Feynman rules.
$\Rightarrow$ the virtual particles of the propagator yields an imaginary part $i\epsilon$ if they go *on-shell*.
Let's check

$$-i(\mathcal{M}(k_1k_2 \to p_1p_2) - \mathcal{M}^*(k_1k_2 \to p_1p_2)) = \sum_f \int \mathrm{d}\Pi_f \mathcal{M}(p_1p_2 \to \{q_f\})\mathcal{M}^*(k_1k_2 \to \{q_f\}) \cdot (2\pi)^4\delta^{(4)}(k_1 + k_2 - \sum_f q_f)$$

for ϕ^4-theory diagrams
threshold energy s_0 is need for production of the lightest multi-particle state ($s = E^2_{\mathrm{CM}}$, Mandelstam-variable).

The Optical Theorem for Feynman Diagrams

properties of $\mathcal{M}$

$$\begin{aligned} \mathcal{M}(s) &= [\mathcal{M}(s^*)]^* \;\; s < s_0 \; (\mathcal{M}(\mathrm{s}) \text{ analytic!}) \\ \Rightarrow \mathrm{Re}\, \mathcal{M}(\mathrm{s}+\mathrm{i}\epsilon) &= \mathrm{Re}\, \mathcal{M}(\mathrm{s}-\mathrm{i}\epsilon), \;\; \mathrm{s} > \mathrm{s}_0 \\ \mathrm{Im}\, \mathcal{M}(\mathrm{s}+\mathrm{i}\epsilon) &= -\mathrm{Im}\, \mathcal{M}(\mathrm{s}-\mathrm{i}\epsilon), \;\; \mathrm{s} > \mathrm{s}_0 \Rightarrow \; \text{discontinuity} \end{aligned}$$

Attention!

- ϕ^4-theory
- $\Rightarrow$ the simplest diagram in our case is a one loop diagram (order $\propto \lambda^2, \;\; s_0 = 2m$)
- the generalization of the result for multi-loop diagrams has been proven by *Cutkosky*
- $\Rightarrow$ Cutting Rules

The Optical Theorem for Feynman Diagrams

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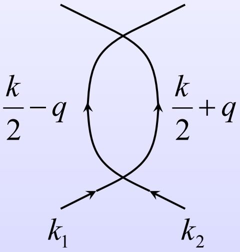
The Optical Theorem for Feynman Diagrams

Consider the one-loop diagram



$$k = k_1 + k_2$$

The Optical Theorem for Feynman Diagrams

Loop-correction

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(k/2-q)^2 - m^2 + i\epsilon}\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \quad (14)$$

Some properties of $\delta\mathcal{M}$:

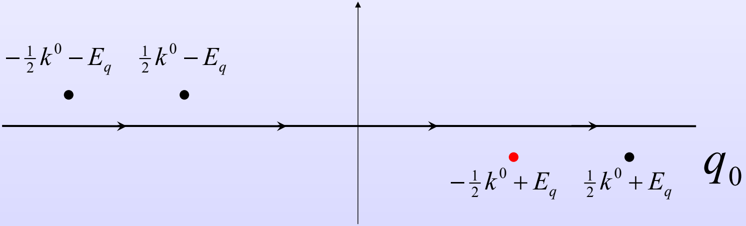
- For $k_0 < 2m$ the integral can be calculated and than we increasing k_0 by analytical continuation
- !! We want to verify, that the integral has a discontinuity across the real axis for $k_0 > 2m$!!

$\Rightarrow$ go into the CM-system $k = (k_0, 0)$

The Optical Theorem for Feynman Diagrams

$\Rightarrow$ we obtain four poles ($E_q^2 = |\mathbf{q}|^2 + m^2$)

$$q^0 = \frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon), \;\; q^0 = -\frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon)$$



$\Rightarrow$ only the pole at $q_0 = -\frac{1}{2}k^0 + E_{\mathbf{q}} - i\epsilon$ will contribute to the discontinuity (close the integration contour in the lower half plane!) $\Rightarrow$ replace: $\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \rightarrow -2\pi i\delta((k/2+q)^2 - m^2)$ under the dq_0-integral

The Optical Theorem for Feynman Diagrams

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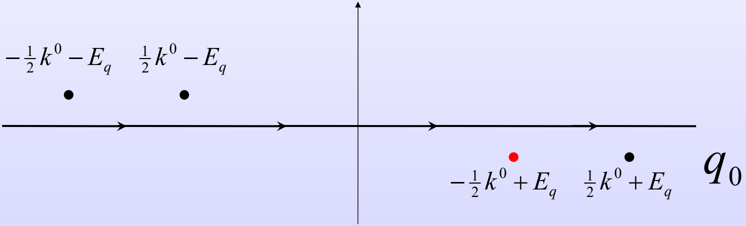

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The Optical Theorem for Feynman Diagrams

$$i\delta\mathcal{M} = -2\pi i\frac{\lambda^2}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^4}\frac{1}{2E_{\mathbf{q}}}\frac{1}{(k^0-E_{\mathbf{q}})^2-E_{\mathbf{q}}^2} \qquad (15)$$

$$= -2\pi i\frac{\lambda^2}{2}\frac{4\pi}{(2\pi)^4}\int_m^\infty \mathrm{d}E_{\mathbf{q}}E_{\mathbf{q}}|\mathbf{q}|\frac{1}{2E_q}\frac{1}{k^0(k^0-2E_{\mathbf{q}})} \qquad (16)$$

properties

- $E_{\mathbf{q}} = k^0/2$ is a pole of the integrand
- if $k^0 < 2m$ the pole doesn't lie in the integration contour $\Rightarrow \mathcal{M}$ is real
- if $k^0 > 2m$ the pole does lie in the integration contour $\Rightarrow k^0$ has a small positive or negative imaginary part

The Optical Theorem for Feynman Diagrams

$$i\delta\mathcal{M} = -2\pi i \frac{\lambda^2}{2} \int \frac{\mathrm{d}^3 q}{(2\pi)^4} \frac{1}{2E_{\mathbf{q}}} \frac{1}{(k^0 - E_{\mathbf{q}})^2 - E_{\mathbf{q}}^2} \quad (15)$$

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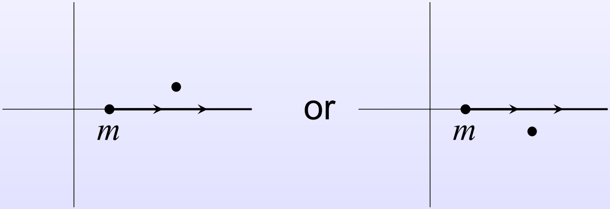
The Optical Theorem for Feynman Diagrams

Integration contour:

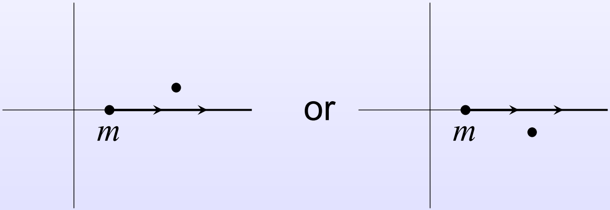


$\Rightarrow$ Thus, the integral has a discontinuity between $k^2 + i\epsilon$ and $k^2 - i\epsilon$!

$$\frac{1}{k^0 - 2E_{\mathbf{q}} \pm i\epsilon} = \mathcal{P}\frac{1}{k^0 - 2E_{\mathbf{q}}} \mp i\pi\delta(k^0 - 2E_{\mathbf{q}}) \tag{17}$$

The Optical Theorem for Feynman Diagrams

Integration contour:



$\Rightarrow$ Thus, the integral has a discontinuity between $k^2 + i\epsilon$ and $k^2 - i\epsilon$!

This is equivalent to replacing the original propagator by a delta-distribution

$$\frac{1}{(k/2-q)^2 - m^2 + i\epsilon} \to -2\pi i \delta((k/2-q)^2 - m^2)$$

The Optical Theorem for Feynman Diagrams - Cutting Rules

Cutting Rules:

Look again at

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(k/2-q)^2-m^2+i\epsilon}\frac{1}{(k/2+q)^2-m^2+i\epsilon}$$

and relabel the momenta at the two propagators with p_1 and p_2.

$\Rightarrow p_1 = k/2 - q,\ \ p_2 = k/2 + q.$

1. Replace:

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} = \iint \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\frac{\mathrm{d}^4 p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k)$$

$$\Rightarrow i\delta\mathcal{M} = \frac{\lambda^2}{2}\iint \frac{\mathrm{d}^4 p_1}{(2\pi)^4}\frac{\mathrm{d}^4 p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k) \times \frac{1}{p_1^2-m^2+i\epsilon}\frac{1}{p_2^2-m^2+i\epsilon}$$

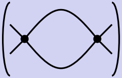
The Optical Theorem for Feynman Diagrams - Cutting Rules

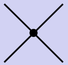
2. Replace:

$$\frac{1}{p_i^2 - m^2 + i\epsilon} \to -2\pi i \delta(p_i^2 - m^2)$$

This gives us in order $\lambda^2 = |\mathcal{M}(k)|^2$

$$2i\mathrm{Im}\ \delta\mathcal{M}(k) = \frac{i}{2}\int \frac{\mathrm{d}p_1^3}{(2\pi)^3}\frac{1}{2E_1}\frac{\mathrm{d}p_2^3}{(2\pi)^3}\frac{1}{2E_2}|\mathcal{M}(k)|^2(2\pi)^4\delta(p_1+p_2-k) \qquad (18)$$

$2i\,\mathrm{Im}\Big($  $\Big)$

$= \frac{i}{2}\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\Big|$  $\Big|^2 \delta^{(4)}(p_1+p_2-k)$

this verifies the optical theorem to order λ^2 in ϕ^4-theory

The Optical Theorem for Feynman Diagrams - Cutting Rules

Cutting Rules

1. Cut through the diagram in *all possible* ways such that the cut propagator can simultaneously be put *on shell*.
2. For each cut, replace $1/(p^2 - m^2 + i\epsilon) \to -2\pi i\delta(p^2 - m^2)$ in each cut propagator, then perform the loop integrals.
3. Sum the contributions of all possible cuts.

Cutkosky proved this method in general.
Using these cutting rules, it is possible to check the optical theorem for all orders in perturbation theory.

The Optical Theorem for Feynman Diagrams - Cutting Rules

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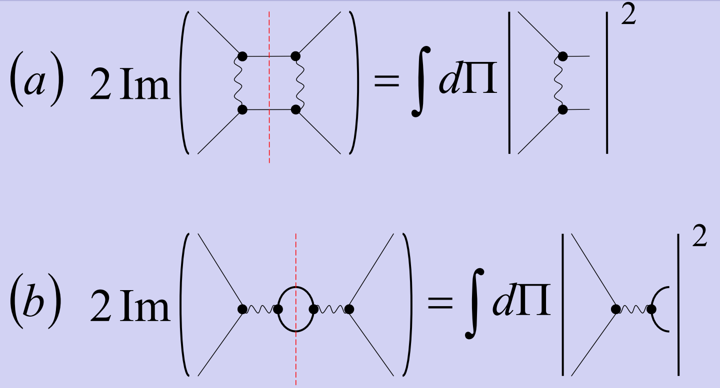
The Optical Theorem for Feynman Diagrams - Example

Bhabha-scattering:



Two contributions to the optical theorem for Bhabha-scattering.

Partial Wave Unitarity

Partial Wave Unitarity:

For the $\mathcal{M}$-Matrix we have found

$$-i(\mathcal{M}(k_1k_2 \to p_1p_2) - \mathcal{M}^*(p_1p_2 \to k_1k_2))$$
$$= \sum_n \left(\prod_{f=1}^{n} \int \frac{\mathrm{d}^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \mathcal{M}(p_1p_2 \to \{q_f\})\mathcal{M}^*(k_1k_2 \to \{q_f\})$$
$$\cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f) \qquad (19)$$

Choose particle k_2 at rest

Consider spinless particles $\Rightarrow$ ϕ-independence

$\Rightarrow$ scattering angle $\theta \Rightarrow \mathcal{M}(k_1k_2 \to p_1p_2) = \mathcal{M}_{ij}(s, \theta)$ where $i = k_1 + k_2$ and $j = p_1 + p_2$ denotes the *initial-* and *final-state*.

Partial Wave Unitarity

scattering amplitude

$$\Rightarrow \mathcal{M}_{ij}(s,\theta) \equiv 8\pi s^{1/2} f_{ij}(s,\theta) \quad (20)$$

$$= 8\pi s^{1/2} \sum_{l=0}^{\infty} (2l+1)\mathcal{M}_{ij,l}(s) P_l(\cos\theta) \quad (21)$$

with $f_{ij}(s,\theta)$ as the scattering amplitude.
If we put $f_{ii}(s,0) \equiv f(0)$

Prove: Optical Theorem

$$\mathrm{Im\ f}(0) = \frac{|\mathbf{P}_{\mathrm{CM}}|}{4\pi}\sigma_{\mathrm{tot}} \quad (22)$$

Partial Wave Unitarity

Two-Particle Partial Wave Unitarity:
If we consider only elastic scattering (i=j)

$$\mathrm{Im}\ \mathcal{M}_{\mathrm{l}} = \sum_{\mathrm{k}} \mathrm{p_k} |\mathcal{M}_{\mathrm{k,l}}|^2 \tag{23}$$

and that all k-channels are closed at low energies ($p_k = p$)

$$\mathrm{Im}\ \mathcal{M}_{\mathrm{l}} = \mathrm{p} |\mathcal{M}_{\mathrm{l}}|^2 \tag{24}$$

$$\Rightarrow \mathcal{M}_l = \frac{1}{p} e^{i\delta_l} \sin \delta_l$$

where δ_l denotes the scattering-phase for the l-th partial wave

Partial Wave Unitarity

The differential cross-section is in general given by

$$\frac{\mathrm{d}\sigma_{\mathrm{ij}}}{\mathrm{d}\Omega} = \frac{1}{16\pi^2}\frac{p'}{p}\frac{1}{4s}\left|\mathcal{M}_{ij}(s,\theta)\right|^2 \tag{25}$$

Using

$$\mathcal{M}_{ij}(s,\theta) = 8\pi s^{1/2}\sum_{l=0}^{\infty}(2l+1)\mathcal{M}_{ij,l}(s)P_l(\cos\theta)$$

Partial Wave Unitarity

we get

$$\sigma_{ij} = 4\pi \frac{p'}{p} \sum_l (2l+1) |\mathcal{M}_{ij,l}|^2 \equiv \sum_l \sigma_{ij,l} \tag{26}$$

For pure elastic scattering at low energies

Partial total cross-section

$$\sigma_l = \frac{4\pi}{p^2} (2l+1) \sin^2 \delta_l \tag{27}$$

Partial Wave Unitarity

And for the case: A and B carry spin

Partial total cross-section for spin 1/2 particles

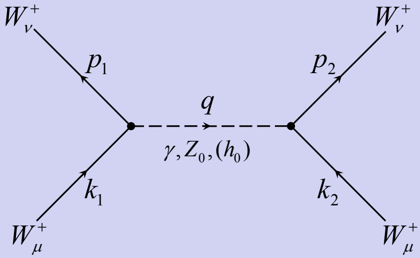
$$\sigma_j = 4\pi \frac{2j+1}{(2s_1+1)(2s_2+1)} \sum_{\lambda_1'\lambda_2'\lambda_1\lambda_2} \left|\mathcal{M}^j(\lambda_1'\lambda_2';\lambda_1\lambda_2;s)\right|^2 \qquad (28)$$

we λ_i are the initial and λ_i' the final helicities

Partial Wave Unitarity - Outlook

short outlook: W^+W^+-scattering $\Rightarrow$ need higgs-boson!

W^+W^+-scattering



Partial Wave Unitarity - Outlook

It is possible to show

$$-i\mathcal{M}_{\gamma+Z^0+\times}(s) = -ig^2\left[\left(\frac{s}{M_W^2}\right)+2\right] \propto s \tag{29}$$

but from the optical theorem follows for an large s

$$|\mathcal{M}(s)| < 16\pi\frac{q^2}{t_0}(\ln s)^2 \text{ (result by [ItZu])} \tag{30}$$

$\Rightarrow$ there must be a counter-term in eq. (29) to cancel the s

Partial Wave Unitarity - Outlook

put in the Higgs-Boson h_0

$$-i\mathcal{M}(s) = -i(\mathcal{M}_{\gamma+Z^0+\times}(s) + \mathcal{M}_{h_0}(s)) = -ig^2\left[4 + \frac{1}{2}\left(\frac{M_{h_0}}{M_W}\right)^2\right] \tag{31}$$

$\Rightarrow$ OK!

Summary

1. We have proved, that the optical theorem follows directly from the *unitarity* of the *S-matrix*
2. Proving the optical theorem for *Feynman diagrams* in ϕ^4-theory we had found the cutting rules
3. We have derived an equation for the partial total cross-section for bosonic and fermionic particles from the principles of the optical theorem

Summary

1. We have proved, that the optical theorem follows directly from the *unitarity* of the *S-matrix*
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