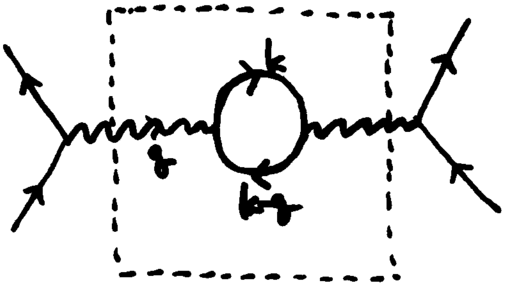
Renormalization Group Flow

1. Introduction : Renormalization



$$i\Pi^{\mu\nu}(q) = -e^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left(\gamma^\mu \frac{\not{k}+m}{k^2-m^2+i\epsilon}\, \gamma^\nu \frac{\not{k}+\not{q}+m}{(k+q)^2-m^2+i\epsilon}\right)$$

: logarithmic divergent

$$\alpha_{eff}(q^2) \simeq \frac{\alpha_0}{1-\frac{\alpha_0}{3\pi}\log\left(\frac{m^2-q^2}{\Lambda^2}\right)} \quad ; \ q^2 \ll 1$$

$$\alpha_{eff}(q^2=0) = \alpha = \frac{1}{137} = \frac{\alpha_0}{1-\frac{\alpha_0}{3\pi}\log\left(\frac{m^2}{\Lambda^2}\right)}$$

$$\Rightarrow \quad \alpha_{eff}(q^2) = \frac{\alpha}{1-\frac{\alpha}{3\pi}\log\left(\frac{m^2-q^2}{m^2}\right)} \quad , \ \Lambda \to \infty$$

: α_0, Λ are disappeared when we express $\alpha_{eff}(q^2)$ using $\alpha_{eff}(q^2=0)$, an observable quantity.

2. Systematic Approach

2.1. QED as a Renormalizable QFT

- Superficial degree of divergence

P_e, P_γ = (number of propagators)

N_e, N_γ = (number of external lines)

V = (number of vertices) $= 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e)$

L = (number of loop integral) $= P_e + P_\gamma - V + 1$

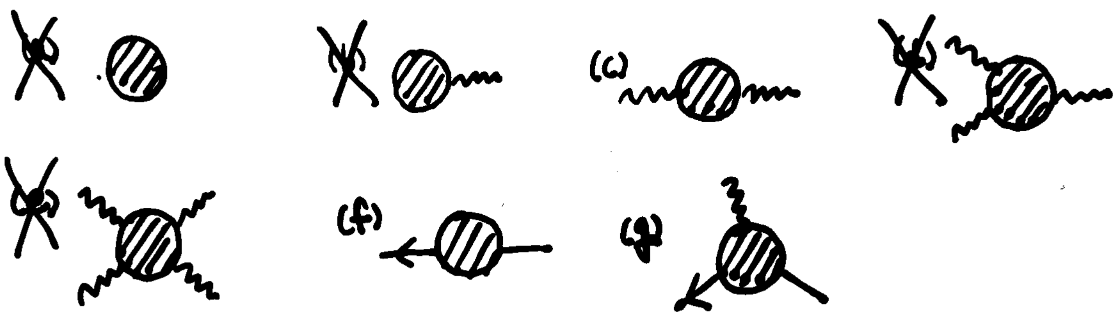
$D \equiv$ (power of k in numerator) $-$ (power of k in denominator)

$$= 4L - P_e - 2P_\gamma$$

$$= 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma = 4 - N_\gamma - \frac{3}{2}N_e$$

$\Rightarrow$ - D depends only on the number of external legs

- A small number of external legs have $D > 0$:



In general,

$$D \equiv dL - P_e - 2P_\gamma = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e$$

$$\begin{cases} d<4 & : \text{Super-renormalizable} \\ d=4 & : \text{Renormalizable} \\ d>4 & : \text{Non-renormalizable} \end{cases}$$

2.2. ϕ^n Theory

- ϕ: scalar field in d-dim.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$$

$$L = P - V + 1$$

n lines meeting at each vertex : $nV = N + 2P$

$$\Rightarrow D = dL - 2P$$

$$= d + \left[n\left(\frac{d-2}{2}\right) - d \right] V - \left(\frac{d-2}{2}\right) N$$

$\Rightarrow$ $d=4$: $n=4$ renormalizable

$n=3$ ~~non~~renormalizable

$d=3$: $n=6$ renormalizable

$n=4$ super-renormalizable

$d=2$: $\forall n$, super-renormalizable

- **Dimensional analysis**

$$S = \int d^d x \, \mathcal{L}$$

$$[\mathcal{L}] = d$$

$$[\phi] = \frac{d}{2} - 1 = \frac{d-2}{2}$$

$$[m] = 1$$

$$[\lambda] = d - \frac{n(d-2)}{2} = -(\text{coefficient of } V)$$

$\Rightarrow$ (super-renormalizable) $\Leftrightarrow$ ([coupling constant] > 0)

3. Renormalization Group Flow

3.1. Flow of Electromagnetic Coupling

Recall that $\alpha = \frac{e^2}{4\pi}$ and take $q^2 = \mu^2$

$$e_{eff}(\mu^2) = e^2 \frac{1}{1+e^2\pi(\mu^2)}$$

$m_e \ll \mu \ll \Lambda$,

$$\mu \frac{d}{d\mu} e_{eff}(\mu) = -\frac{1}{2} e^3 \mu \frac{d}{d\mu} \pi(\mu^2) + O(e^5) = \frac{1}{12\pi^2} e_{eff}^3 + O(e^5) > 0$$

In general, in a QFT with a coupling constant g,

$$\mu \frac{dg}{d\mu} = \beta(g)$$: The rate of change of the renormalized coupling at the scale μ

Defining $t \equiv \log \frac{\mu}{\mu_0}$

$$\frac{dg_i}{dt} = \beta_i (g_1, \cdots, g_N)$$

g^* such that $\beta_i(g^*) = \left.\frac{dg}{dt}\right|_{g=g^*} = 0$: fixed point.

3.2. Wilson's Approach

$$\mathcal{L} = (\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

In Euclidean space,

$$Z(\Lambda) = \int_\Lambda D\phi \, \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]\right) ;$$

where

$$\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx}\phi(k) \qquad \text{such that} \quad \phi(k) = 0 \; ; \; |k| > \Lambda$$

- Integrating over high-momentum DoF such that

$b\Lambda \le |k| < \Lambda$ with $b < 1$

$$\left.\begin{array}{l} \hat{\phi}(k) \equiv \begin{cases} \phi(k) , & b\Lambda \le |k| < \Lambda \\ 0 , & \text{otherwise} \end{cases} \\ \\ \phi(k) \equiv \begin{cases} 0 , & |k| \ge b\Lambda \\ \phi(k) , & |k| < b\Lambda \end{cases} \end{array}\right\} \quad \phi \to \phi + \hat{\phi}$$

$$Z(\Lambda) = \int D\phi \int D\hat{\phi} \, \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\phi + \partial_\mu\hat{\phi})^2 + \frac{1}{2}m(\phi+\hat{\phi})^2 + \frac{\lambda}{4!}(\phi+\hat{\phi})^4\right]\right)$$

$$= \int D\phi \, e^{-\int\mathcal{L}(\phi)} \int D\hat{\phi} \, \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\right)$$

$$= \int_{b\Lambda} D\phi \, \exp\left(-\int d^dx \, \mathcal{L}_{eff}\right)$$

- $\mathcal{L}_{eff}$ involves only $\phi(k)$ with $|k| < b\Lambda$
- $\mathcal{L}_{eff}(\phi) = \mathcal{L}(\phi) + (\text{corrections proportional to powers of } \lambda)$

- Rescaling $k' = k/b$; $x' = xb$

$$\int d^d x\, \mathcal{L}_{\text{eff}} = \int d^d x \Big[\tfrac{1}{2}(1+\Delta Z)(\partial_\mu \phi)^2 + \tfrac{1}{2}(m^2+\Delta m^2)\phi^2 + \tfrac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C(\partial_\mu\phi)^4 + \Delta D \phi^6 + \cdots \Big]$$

$$= \int d^d x'\, b^{-d} \Big[\tfrac{1}{2}(1+\Delta Z)\, b^2 (\partial'_\mu \phi)^2 + \tfrac{1}{2}(m^2+\Delta m^2)\phi^2 + \tfrac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C\, b^4 (\partial'_\mu\phi)^4 + \Delta D \phi^6 + \cdots \Big]$$

$$\phi' = \left[b^{2-d}(1+\Delta Z) \right]^{1/2} \phi$$

$$\Rightarrow \int d^d x\, \mathcal{L}_{\text{eff}} = \int d^d x' \Big[\tfrac{1}{2}(\partial'_\mu \phi')^2 + \tfrac{1}{2} m'^2 \phi'^2 + \tfrac{1}{4!}\lambda' \phi'^4 + C'(\partial'_\mu\phi')^4 + D'\phi'^6 + \cdots \Big]$$

with

$$m'^2 = (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2}$$

$$\lambda' = (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4}$$

$$C' = (C + \Delta C)(1+\Delta Z)^{-2} b^{d}$$

$$D' = (D + \Delta D)(1+\Delta Z)^{-3} b^{2d-6}$$

(Integrating out high-momentum DOF) + (Rescaling)

$\equiv$ (transformation of $\mathcal{L}$)

As $b \to 1$, the shells of momentum space $\longrightarrow$ thinner

transformation of $\mathcal{L}$ $\longrightarrow$ continuous

$\Rightarrow$ Renormalization group : continuously generated transformations of $\mathcal{L}$

- $\mathcal{L}$ Vs. $\mathcal{L}_{\text{eff}}$?

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$$

$$\begin{cases} m'^2 = m^2 b^{-2} & \uparrow \quad (\phi^2 : \text{relevant}) \\ \lambda' = \lambda b^{d-4} & \uparrow,\ d<4 \quad (\phi^4 : \text{relevant}) \ ; \downarrow, d>4 \ (\text{irrelevant}) \ ; \ d=4 \ (\text{marginal}) \\ c' = c b^{d} & \downarrow \\ D' = D b^{2d-6} & \uparrow,\ d<3 \end{cases}$$

In general,

$$C'_{N,M} = b^{-d}(b^{2-d})^{-N/2} b^{M} = b^{\underbrace{N(d/2-1)+M}_{d_i}-d} C_{N,M}$$

$$\Rightarrow [\lambda] = d - d_i \quad ; \quad \begin{cases} d_i < d & : \text{relevant} \\ d_i > d & : \text{irrelevant.} \end{cases}$$

* In the vicinity of the zero-coupling fixed point, an arbitrary complicated $\mathcal{L}$ at the scale of cutoff degenerates to a $\mathcal{L}$ containing only a finite number of renormalizable interactions

- Regularization with $\Lambda \to \infty$ Vs. Wilson's approach

- Example : ϕ^4 theory

i) $d > 4$

$m'^2 = m^2 b^{-2n} \quad \Rightarrow \exists n$ such that $m'^2 \sim \Lambda^2$

$m \ll \Lambda \quad \Leftrightarrow \quad m'^2 \sim \Lambda^2 \ , \ n \gg 1$

$\Leftrightarrow$ The initial condition is set so that the trajectory passes very close to a fixed point.

$\Rightarrow$ (complicated nonlinear $\mathcal{L}$) $\rightarrow$ (simple effective $\mathcal{L}$)

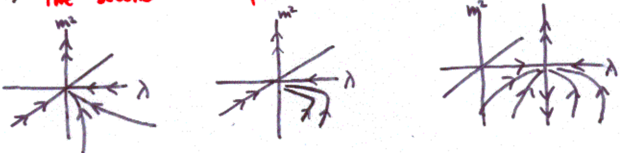
ii) $d=4$

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log\left(\frac{1}{b}\right)$$

iii) $d<4$

$$\lambda' = \lambda b^{d-4}$$

$\Rightarrow$ The second fixed point



3.3 The Callan–Symanzik Equation

M: renormalization scale

$$G^{(n)}(x_1,\cdots,x_n) = \langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle_{\text{connected}}$$

$$M \longrightarrow M+\delta M$$
$$\lambda \longrightarrow \lambda+\delta\lambda$$
$$\phi \longrightarrow (1+\delta\eta)\phi \quad \Rightarrow \quad G^{(n)} \longrightarrow (1+n\delta\eta)G^{(n)}$$

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda = n\delta\eta\, G^{(n)}$$

$$\Rightarrow \beta \equiv \frac{M}{\delta M}\delta\lambda \ ; \ \gamma \equiv -\frac{M}{\delta M}\delta\eta$$

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right]G^{(n)}(\{x_i\};M,\lambda)=0$$

: Callan–Symanzik equation.

In terms of the parameters of bare perturbation theory;

$$\phi(p) = Z(M)^{-1/2}\phi_0(p)$$

$$\delta\eta = \frac{Z(M+\delta M)^{-1/2}}{Z(M)^{-1/2}} - 1$$

$$\Rightarrow \gamma(\lambda) = \frac{1}{2}\frac{M}{Z}\frac{\partial}{\partial M}Z \ ; \ \beta(\lambda) = M\frac{\partial}{\partial M}\lambda\Big|_{\lambda_0,\Lambda}$$

For the two-point Green's function, $G^{(2)}(p) = \frac{i}{p^2}g(-p^2/M^2)$,

$$\left[p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2 - 2\gamma(\lambda)\right]G^{(2)}(p) = 0 \ , \quad \text{where } p \to (-p^2)^{1/2}$$

3.4. Evolution of Coupling Constants

$v(x)$: fluid velocity in a narrow pipe

$D(t;x)$: density of bacteria

$\rho(x)$: growth rate of bacteria.

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - \rho(x)\right] D(t;x) = 0$$

$$\left\{\begin{array}{l} \log(p/M) \longleftrightarrow t \\ \lambda \longleftrightarrow x \\ -\beta(\lambda) \longleftrightarrow v(x) \\ 2\gamma - 2 \longleftrightarrow \rho(x) \\ G^{(n)}(p,\lambda) \longleftrightarrow D(t;x) \end{array}\right.$$

$\bar{x}(t;x)$: position of fluid element that is at x, t, at $t=0$

$$\frac{d}{dt'}\bar{x}(t';x) = -v(\bar{x}) \quad , \quad \text{with} \quad \bar{x}(0;x) = x.$$

$$\Rightarrow D(t;x) = D_i(\bar{x}(t;x)) \cdot \exp\left(\int_0^t dt' \rho(\bar{x}(t';x))\right) = D_i(\bar{x}(t;x)) \cdot \exp\left(\int_{\bar{x}(t)}^{x} dx' \frac{\rho(x')}{v(x')}\right)$$

$$\Leftrightarrow G^{(n)}(p,\lambda) = \hat{G}(\bar{\lambda}(p;\lambda)) \cdot \exp\left(-\int_{p'=M}^{p'=p} d(\log(p'/M)) \cdot 2[1-\gamma(\bar{\lambda}(p';\lambda))]\right),$$

where $\dfrac{d}{d\log(p/M)}\bar{\lambda}(p;\lambda) = \beta(\bar{\lambda})$; $\bar{\lambda}(M;\lambda) = \lambda$

: renormalization group equation

$\bar{\lambda}(p)$: running coupling constant.

4. Summary

- From the mass dimension of the coupling constant, we can directly see the renormalizability of the theory : QED is renormalizable.

- We can describe the renormalization process as the flow in the space of possible Lagrangians
  : Renormalization group flow

- QFT is an effective low energy theory valid up to some energy scale

- The solution of the Callan-Symanzik equation depends on a running coupling constant which satisfies the renormalization group equation.