Solitons, Vortices, Instantons

Seminarvortrag von
David Mroß
Universität Bonn

1. Juni 2006

Structure of the talk

- Solitons in a nutshell
- Recap: solitons in $1+1$ dim space-time
- Intermezzo: topological mappings
- $2+1$: vortices
  - vortex-antivortex
  - gauging for finite energy
- $3+1$: the hedgehog
- $(d)+(1) \leftrightarrow (d+1)$: the instanton
  - the instanton as a soliton
  - interpretation: vacuum tunneling
  - instantons and gauge theory
- Summary

Solitons in a nutshell

- Solitons were first discovered in the 19th century as surface waves on water.
- Whereas usually localized waves change their shape due to dispersion, solitons do not.
- mathematically, solitons can appear in nonlinear diff. equations, the nonlinearity compensates the dispersion.
- Typical (integrable) examples for nonlinear equations where solitons appear are:
  - the KdV-equation $\partial_t \psi + \partial_x^3 \psi + 6\psi \partial_x \psi = 0$, e.g. in hydrodynamics
  - the nonlinear-Schrödiner equation $\mathrm{i}\partial_t \psi + \partial_x^2 \psi \mp 2|\psi|^2 \psi = 0$, e.g. in waveguides
  - The sine-gordon equation, $\partial_x^2 \psi - \partial_t^2 \psi = \sin \psi$, e.g. in condensed matter

Recap: solitons in $1+1$ dim space-time

In one of the previous talks, we were looking for nontrivial, stationary finite energy solutions (to the e.o.m.) in $1+1$ dim. space-time with

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2,$$

(for a real, scalar field ϕ). The energy is

$$M = \int dx \left(\frac{1}{2}\left(\frac{d\phi}{dx}\right) + \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2\right),$$

thus $\quad \phi(r \to \infty) = \pm v.$

Trivial solutions are the vacua ($M = 0$):

$$\phi_\pm(x) = \phi_\pm(+\infty) = \phi_\pm(-\infty) = \pm v.$$

Recap: solitons in $1+1$ dim space-time

We found the kink and antikink with $M \sim \mu v^2$ ($\mu^2 = \lambda v^2$), concentrated in a region $l \sim \frac{1}{\mu}$.

$$J_{top} := \frac{1}{2v} \epsilon^{\mu\nu} \partial_\nu \phi$$

leads to the conserved charge

$$Q_{top} = \int_{-\infty}^{-\infty} dx J^0_{top}(x) = \frac{1}{2v} \left(\phi(+\infty) - \phi(-\infty)\right) \in \mathbb{Z},$$

which implies that kink ($Q_{top} = 1$) and antikink ($Q_{top} = -1$) are stable with respect to the vacuum ($Q_{top} = 0$).

Intermezzo: topological mappings

The asymptotic condition,

$$\phi(r \to \infty) = \pm v,$$

can be described as a mapping of the 2-point-set $(-\infty, \infty)$ to the 2-point-set $(-v, v)$, which is equivalent to a mapping $S^0 \to S^0$.

Definition: Two continous mappings g and f from X to Y (topological spaces) are *homotopic* if there exists a continuous function (*homotopy*) $H(t, x) : [0, 1] \times X \to Y$ with $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

Intermezzo: topological mappings

In our cases we are always interested in mappings between spheres, i.e. $\phi_\infty : S^n \longrightarrow S^n$ as spatial infinity in $\mathbb{R}^n$ is topologically equivalent to $S^{(n-1)}$

It can be shown that for $n \geq 1$ such mappings can be characterized by an integer winding number ($n \in \mathbb{Z}$) which is called the Pontryargin index.

Consider for example $S^1 \rightarrow S^1$. The functions

$$f_{a,n}(\theta) = \exp \imath(n\theta + a) \qquad \theta \in [0, 2\pi]$$

for fixed integer n and arbitrary a are homotopic with

$$H_{a,b,n}(t,\theta) = \exp \imath \left(n\theta + (1-t)\, a + tb\right).$$

2+1: Vortices

Consider a complex scalar field with

$$\mathcal{L} = \partial\phi^\dagger\partial\phi - \lambda\left(\phi^\dagger\phi - v^2\right)^2 .$$

The Mass of a soliton would be

$$M = \int d^2x \left[\partial_i\phi^\dagger\partial_i\phi + \lambda\left(\phi^\dagger\phi - v^2\right)^2\right].$$

Finiteness requires $|\phi| \longrightarrow v$ at spatial infinity.
This suggests the ansatz $\phi(r \to \infty) = ve^{i\theta}$ in polar coordinates.

2+1: Vortices, the picture

Writing $\phi = \phi_1 + \mathrm{i}\phi_2$ we see that
$(\phi_1, \phi_2) \to v(\cos\theta, \sin\theta) = \frac{v}{r}(x, y)$.
This vector points radially outwards.

Computing the gradient of the field,

$$\vec{\partial}\phi \to \frac{v}{r}\begin{pmatrix}\partial_x\,(x + \imath y)\\ \partial_y\,(x + \imath y)\end{pmatrix} = \frac{v}{r}\begin{pmatrix}1\\ -\imath\end{pmatrix}$$

leads to the current:

$$\vec{J} = \imath\left(\vec{\partial}\phi^\dagger\phi - \phi^\dagger\vec{\partial}\phi\right) \to \frac{v}{r}\begin{pmatrix}-2y\\ 2x\end{pmatrix}.$$

This is a vector of constant length, $2v$ whirling around at spatial infinity.

2+1: Vortices

Obviously ϕ maps $S^1 \to S^1$, thus we know the homotopy group to be $\mathcal{Z}$ and vortices are topologically stable with respect to the vacuum. $[\phi]^n$ has the same properties and we identify n as the conserved charge (winding number).

Plugging the behavior of ϕ, i.e. $\phi \sim v$ and $\partial_i \phi \sim \frac{v}{r}$, into the expression for the energy yields:

$$M = \int d^2x \left[\partial_i \phi^\dagger \partial_i \phi + \lambda \left(\phi^\dagger \phi - v^2 \right)^2 \right] \sim v^2 \int d^2x \frac{1}{r^2}.$$

This is, of course, logarithmically divergent.

What can we do?

- consider vortex-antivortex pairs
- gauge the theory

2+1: Two vortices

Now consider a vortex and an antivortex (vortex with negative charge) at some (large) distance $R = R_1 - R_2$ ($R \gg a$ where a is the size of the vortex):

$$\varphi = \phi_+(r + R_2)\phi_-(r + R_1).$$

At infinity $\varphi \to v$ and $\partial\varphi \to 0$ thus M is finite.
Between R_1 and R_2, we find $\varphi \sim ve^{2\imath\theta}$.
Now a very rough estimate of the energy is

$$M \sim v^2 \int d^2x \frac{1}{r^2} \sim v^2 \log \frac{R}{a}.$$

We have an attractive $\log$ potential (as in 2-dim. coulomb case). This configuration cannot be static, vortex and antivortex tend to annihilate and release energy.

2+1: Gauging for finite energy

We gauge the theory in the usual way by replacing $\partial_i \phi$ with $D_i \phi = \partial_i \phi - \imath e A_i \phi$. Then finite energy can be achieved by requiring

$$A_i(r \to \infty) \longrightarrow -\frac{\imath}{e} \frac{1}{|\phi|^2} \phi^\dagger \partial_i \phi = \frac{1}{e} n \partial_i \theta$$

where n is the winding number.

$$\Rightarrow D_i \phi \to \partial_i \phi - \frac{\phi^\dagger \partial_i \phi}{|\phi|^2} \phi = \partial_i \phi (1 - \frac{\phi^\dagger \phi}{|\phi|^2}) = 0$$

We can then also calculate the flux as

$$Flux \equiv \int d^2x \; B = \oint_C dx_i A_i = \frac{n}{e} \oint_C dx_i \frac{d}{dx_i} \theta = \frac{n 2\pi}{e}.$$

This vortex appears as a flux tube in type II superconductors.

3+1: The hedgehog

In $3+1$ dimensions we proceed as before. Spatial infinity now is S^2, thus we consider scalar fields $\phi_a(a = 1, 2, 3)$, transforming as a vector $\vec{\phi}$ under $O(3)$ with the Lagrangian $\mathcal{L} = \frac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} - \lambda\left(\vec{\phi}^2 - v^2\right)^2$. The energy (time-independent) is then:

$$M = \int d^3x \left[\frac{1}{2}\left(\partial\vec{\phi}\right)^2 + \lambda\left(\vec{\phi}^2 - v^2\right)^2\right]$$

Again we have the requirement $|\vec{\phi}(r \to \infty)| \to v$, thus $\vec{\phi}(r = \infty)$ lives on S^2.
The obvious choice for our fields is now

$$\phi^a(r \to \infty) = v\frac{x^a}{r}.$$

Just like the vortex, this does not yet have finite energy, and we therefore gauge the theory, with an $O(3)$-gauge-potential, A^b_μ.

3+1: The hedgehog, gauging for finite energy

So we replace $\partial_i \phi^a$ by

$$D_i \phi^a = \partial_i \phi^a + e\epsilon^{abc} A_i^b \phi^c$$

and choose A such that $D_i \phi^a$ vanishes at infinity:

$$A_i^b(r \to \infty) = \frac{1}{e}\epsilon^{bij}\frac{x^j}{r^2}$$

If we now consider a small lab at infinity, $\vec{\phi}$ points approximately in the same direction everywhere, $O(3)$ ist broken down to $U(1)$.
The massless gauge field points radially outwards, it can be interpreted as the t'Hooft monopol, which we have learned about earlier.
Its mass has been calculated to be $\sim 137 M_W$

$(d) + (1) \leftrightarrow (d+1)$: The instanton as a soliton

Let us now consider time-dependent configurations in d space and 1 time dimensions:

$$\mathcal{S} = \int d^d x dt \left[\mathcal{L}\right] = \int d^d x dt \left[\frac{1}{2}\left(\partial_t \phi\right)^2 - \frac{1}{2}\left(\vec{\partial}\phi\right)^2 - V(\phi)\right]$$

Upon performing a Wick rotation we get the Euclidian action:

$$\mathcal{S}_E = \int d^d x d\tau \left[\frac{1}{2}\left(\partial_t \phi\right)^2 + \frac{1}{2}\left(\vec{\partial}\phi\right)^2 + V(\phi)\right]$$

For the instanton we require $\mathcal{S}_E = \int d^{d+1}x \left[\frac{1}{2}\delta^{ab}\partial_a\phi\partial_b\phi + V(\phi)\right]$ to be finite.
For the solitons we had required $M = \int d^D x \quad \left[\frac{1}{2}\delta^{ab}\partial_a\phi\partial_b\phi + V(\phi)\right]$ to be finite.

Obviously, for $d + 1 = D$ these conditions are equivalent!

Interpretation of the Instanton: Vacuum tunneling

Now we shall take a look at $0+1$ dim. space-time, i.e. ordinary (quantum) mechanics (if we identify ϕ with the coordinate x). Again consider a double-well potential

$$V(\phi) = (\phi^2 - v^2)^2 \qquad \phi \text{ real, scalar.}$$

We know that for imaginary time we can have instanton solutions which are just the kinks from $1+1$ dim. space-time!

$$\mathcal{S}_{E,kink} = \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{2} \left(\frac{\partial \phi_{kink}}{d\tau} \right)^2 + V(\phi_{kink}) \right] = \text{finite}$$

$$\langle -v | e^{-\imath Ht} | v \rangle = \int [d\phi] e^{iS} \longrightarrow \langle -v | e^{-H\tau} | v \rangle = \int [d\phi] e^{-S_E} \neq 0$$

In euclidian space-time we have a finite transition amplitude between the vacua.

Interpretation of the Instanton: Vacuum tunneling

Classically, the ground state (vacuum) is either

$$|vac\rangle = |v\rangle \text{ or } |vac\rangle = |-v\rangle$$

Quantum mechanically the vacuum is

$$|vac\rangle = \frac{1}{\sqrt{2}}\left(|v\rangle + |-v\rangle\right),$$

due to tunneling.
This suggest we interpret the instanton as a tunneling process between different vacua.
Since in the path integral all path are weighed with e^{-S_E}, configurations with finite action, i.e. instantons will dominate.

Instantons and gauge theory

Now, as the final part we consider an nonabelian gauge-theory without scalar fields in euclidian space.

$$\mathcal{S}_E(A) = \int d^4x \frac{1}{2g^2} tr F_{\mu\nu} F_{\mu\nu}$$

with $A_\mu = \frac{\tau^a}{2} A_\mu^a$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

Under a gauge transformation U:

$$A'_\mu = U^{-1} A_\mu U + U^{-1} \partial_\mu U.$$

Obviously, finite action, i.e. an instanton, can be achieved by the pure gauge:

$$A(r \to \infty) = U^{-1} \partial_\mu U.$$

Instantons and gauge theory

Now, for $SU(2)$ we can write

$$U = \exp(\imath\vec{\epsilon}\cdot\vec{\tau}) = u_0 + \imath\vec{u}\cdot\vec{\tau},$$

with real u_0 and $\vec{u}$, that have to satisfy $u_0^2 + \vec{u}^2 = 1$ because U is unitary.
Clearly this is the equation for S^3 (S^3 is the group manifold of $SU(2)$).

$$U : S^3(\text{infinity in euclid. spacetime}) \longrightarrow S^3$$

Now it can be shown that for a mapping $f : S^3 \to S^3$, $h_i := f^{-1}\partial_i f$ the winding number is:

$$n(S^3 \to S^3) = \frac{-1}{24\pi^2}\int d\theta_1 d\theta_2 d\theta_3 tr\left(\epsilon_{ijk}h_i h_j h_k\right) \in \mathbb{Z}$$

Instantons and gauge theory

With the definitions:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} \qquad \text{and} \qquad K_\mu = 4\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[A_\nu\partial_\lambda A_\rho + \frac{2}{3}a_\nu A_\lambda A_\rho\right]$$

we see that $\partial_\mu K_\mu = 2tr\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right]$.
In our case (pure gauge) $K_\mu = \frac{4}{3}\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[A_\nu A_\lambda A_\rho\right]$.

$$\int d^4x \ \mathrm{tr}\left[F_{\mu\nu}\tilde{F}_{\mu\nu}\right] = \frac{1}{2}\int d^4x\, \partial_\mu K_\mu = \frac{1}{2}\int_{S^3} d\sigma_\mu K_\mu = 16\pi^2 n$$

Now take a look at the axial-vector current: $\partial_\mu J_5^\mu = \frac{1}{(4\pi)^2}\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[F_{\mu\nu}F_{\lambda\rho}\right]$

$$Q_5 = Q_R - Q_L = \int d^4x \frac{1}{(16\pi^2)}\epsilon_{\mu\nu\lambda\rho}tr\left[F_{\mu\nu}F_{\lambda\rho}\right] = n$$

Summary

- Kink(1+1 dim): Mechanical model, tunneling (instanton)
- Vortex(2+1 dim): Coulomb gas, flux tubes (gauge theory)
- Hedgehog(3+1 dim): Magnetic monopole
- Instanton(4 dim): Vacuum tunneling, chiral anomaly

Literatur

[1] Ta-Pei Cheng and Ling-Fong Li. *Gauge theory of elementary particle physics*. Springer, 1998.

[2] Bjorn Felsager. *Geometry, Particles, and Fields*. Springer, 1997.

[3] Steudel Meinel, Neugebauer. *Solitonen, Nichtlineare Strukturen*. Akademie Verlag, 1991.

[4] A. Zee. *Quantum field theory in a nutshell*. Princeton University Press, 2003.