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Summer term 2004
Example sheet 1
2004-05-03

Elementary Particle Physics II

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1. ***Calculation tools for Weyl spinors***

- θ_α ($\alpha = 1, 2$) and $\bar\theta^{\dot\alpha}$ ($\dot\alpha = 1, 2$) are anticommuting *Grassmann* variables:

$$\{\theta_\alpha, \theta_\beta\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = \{\theta_\alpha, \bar\theta^{\dot\beta}\} = 0\,. \tag{1}$$

  θ transforms as a *left-handed* ($\frac{1}{2}$,0) Weyl spinor under the Lorentz group, $\bar\theta$ as a *right-handed* (0,$\frac{1}{2}$) one. (*For more background about Weyl spinors, see last term's example sheet 2.*)

- $\epsilon_{\alpha\beta} \equiv \epsilon^{\alpha\beta} \equiv \epsilon_{\dot\alpha\dot\beta} \equiv \epsilon^{\dot\alpha\dot\beta}$ are totally antisymmetric tensors, defined through $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{12} = 1$.

(a) Verify that $\epsilon_{\alpha\beta}\,\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$.

(b) The ϵ are Lorentz invariant and can be used to raise and lower spinor indices. We define

$$\theta_\alpha \equiv \epsilon_{\alpha\beta}\,\theta^\beta\,, \qquad \bar\theta_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,. \tag{2}$$

What is the inverse of these relations?

(c) Verify the following identities (*note the conventions for index contraction*):

$$\xi\psi(\equiv \xi^\alpha\psi_\alpha) = \psi\xi\,, \quad \bar\xi\bar\psi(\equiv \bar\xi_{\dot\alpha}\bar\psi^{\dot\alpha}) = \bar\psi\bar\xi\,, \tag{3}$$

$$\xi_\alpha\xi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\,\xi\xi\,, \qquad \xi^\alpha\xi^\beta = ?\,, \qquad \bar\xi^{\dot\alpha}\bar\xi^{\dot\beta} = -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\xi\bar\xi\,, \qquad \bar\xi_{\dot\alpha}\bar\xi_{\dot\beta} = ?\,. \tag{4}$$

(d) We will write the derivative with respect to a Grassmann variable as $\frac{\partial\theta^\beta}{\partial\theta^\alpha} \equiv \partial_\alpha\theta^\beta = \delta_\alpha^\beta = \partial^\beta\theta_\alpha$ and $\frac{\partial\bar\theta_{\dot\beta}}{\partial\bar\theta_{\dot\alpha}} \equiv \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha} = \bar\partial_{\dot\beta}\bar\theta^{\dot\alpha}$. Note that the product rule must include a negative sign: $\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma$. Why is this neccessary? Check that $\partial^\alpha = \epsilon^{\alpha\beta}\partial_\beta$ (*with the opposite sign to (b)!*). Now check that $\partial^\alpha\partial_\alpha(\theta\theta) = \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 4\,.$

2. ***Pauli matrices and Weyl spinors***

The Pauli matrices can be used to link the spinorial indices $\alpha, \dot\alpha$ to the spacetime index μ (a Lorentz vector x can be written either as x^μ or $x_{\alpha\dot\beta} \equiv x_\mu \sigma^\mu_{\alpha\dot\beta}$). We will use these conventions:

$$\sigma^\mu_{\alpha\dot\beta} = (\mathbf{1}, \vec\sigma)_{\alpha\dot\beta}\,, \qquad \bar\sigma^{\dot\beta\alpha}_\mu = (\mathbf{1}, \vec\sigma)^{\dot\beta\alpha}\,, \qquad \eta^{\mu\nu} = diag(1,-1,-1,-1)\,. \tag{5}$$

(a) Check that the definitions in (5) are consistent with the use of ϵ to raise and lower indices. (*Hint from last term:* $-\epsilon\sigma^\mu\epsilon = (\bar\sigma^\mu)^T$)

(b) Check

$$\sigma^\mu_{\alpha\dot\beta}\,\bar\sigma^{\dot\gamma\delta}_\mu = 2\,\delta^\delta_\alpha\,\delta^{\dot\gamma}_{\dot\beta}\,, \qquad \mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2\,\eta^{\mu\nu}\,. \tag{6}$$

(c) Show

$$V^\mu = \frac{1}{2}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\,V_{\alpha\dot\beta}\,. \tag{7}$$

(d) Verify the following identities:

$$(\bar\xi\bar\sigma^\mu\psi) = -(\psi\sigma^\mu\bar\xi)\,, \tag{8}$$

$$\psi_\alpha\bar\xi_{\dot\beta} = \frac{1}{2}\,\sigma^\mu_{\alpha\dot\beta}\,(\psi\sigma_\mu\bar\xi)\,, \qquad (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)\,. \tag{9}$$

3. ***SUSY generators***

The SUSY algebra is defined through the following relations:

$$\left\{Q_\alpha, \bar Q_{\dot\beta}\right\} = 2\,\sigma^\mu_{\alpha\dot\beta}\,P_\mu\,, \quad [Q_\alpha, P_\mu] = \left[\bar Q_{\dot\alpha}, P_\mu\right] = \{Q_\alpha, Q_\beta\} = \left\{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\right\} = 0 \tag{10}$$

(a) Show that $\left[\theta Q, \bar Q\bar\theta\right] = 2\,\theta\sigma^\mu\bar\theta\,P_\mu$.

(b) Check that

$$P_\mu = i\frac{\partial}{\partial x_\mu} \equiv i\partial_\mu\,, \tag{11}$$

$$Q_\alpha = \partial_\alpha - i\,\sigma^\mu_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\partial_\mu\,, \tag{12}$$

$$\bar Q_{\dot\beta} = -\bar\partial_{\dot\beta} + i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu\,. \tag{13}$$

form a representation of the SUSY algebra by explicitly verifying that they satisfy the (anti-)commutators in (10).