Universität Bonn
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Theoretische Physik

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Example sheet 2
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General Relativity

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1. *Electric current (*)*

The electrical charge and current densities of a collection of charged point particles with positions $\vec{x}_n(t)$ and charges e_n are

$$\vec{J}(\vec{x},t) \equiv \sum_n e_n \, \delta^3(\vec{x} - \vec{x}_n(t)) \, \frac{d\vec{x}_n(t)}{dt} \tag{1}$$

and

$$\varepsilon(\vec{x},t) \equiv \sum_n e_n \, \delta^3(\vec{x} - \vec{x}_n(t)) \, . \tag{2}$$

(a) Write $J^\alpha \equiv \binom{\varepsilon}{\vec{J}}$ as a single expression. Show that J^α transforms correctly as a spacetime four-vector.

(b) Show that J^α is a conserved four-current:

$$\partial_\alpha J^\alpha(x) = 0 \, , \tag{3}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha \equiv (\partial/\partial t, \vec{\nabla})$.

(c) Verify that $Q = \int d^3x J^0(x)$ is time-independent.

2. *Electromagnetism (*)*

Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = \varepsilon \, , \qquad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J} \, , \qquad \vec{\nabla} \cdot \vec{B} = 0 \, , \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \, . \tag{4}$$

To make their properties under Lorentz transformations explicit, we can choose an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ such that $F^{12} = B_3 \, , F^{23} = B_1 \, , F^{31} = B_2$, and $F^{01} = E_1 \, , F^{02} = E_2 \, , F^{03} = E_3$.

(a) Show that

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad \text{and} \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \tag{5}$$

reproduce Maxwell's equations. ($\epsilon^{0123} \equiv +1$)

(b) Verify in the rest frame that

$$f^\mu \equiv \frac{dp^\mu}{d\tau} = e F^\mu{}_\nu \frac{dx^\nu}{d\tau} \tag{6}$$

is the correct equation for the electromagnetic four-force f^μ acting on a charged particle. ($p^\mu = m \, dx^\mu / d\tau$)

3. ***Energy-momentum tensor***

In analogy to the electrical charge and current densities in equations (1) and (2), we can define a charge and current density for the four-momentum p^μ, the *energy-momentum tensor*

$$T^{\mu\nu}(\vec{x},t) \equiv \sum_n p_n^\mu(t)\,\frac{dx_n^\nu(t)}{dt}\,\delta^3(\vec{x}-\vec{x}_n(t)) \tag{7}$$

(a) Show that the energy-momentum tensor is only conserved up to a *force density* G^μ which vanishes for free particles:

$$\partial_\nu T^{\mu\nu} = G^\mu\,. \tag{8}$$

(b) Check that for the electromagnetic forces given in (6), we get $G^\mu = F^\mu{}_\nu\, J^\nu$.

(c) To obtain a conserved energy-momentum tensor, we have forgotten to include the contribution of the electromagnetic field itself:

$$T^{\mu\nu}_{em} \equiv F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,. \tag{9}$$

(*) Write T^{00}_{em} and T^{i0}_{em} in terms of $\vec{E}$ and $\vec{B}$. Do you recognize the expressions?

(d) Show that $\partial_\nu T^{\mu\nu}_{em}$ cancels G^μ from (b) exactly.
(Use (5); the second equation is equivalent to $\partial_\mu F^{\nu\rho} + \partial_\nu F^{\rho\mu} + \partial_\rho F^{\mu\nu} = 0$.)

(e) (*) Show that the total momentum $p^\mu = \int d^3x\, T^{\mu 0}(\vec{x},t)$ is a conserved quantity.
(This is completely analogous to 1.(c))

4. ***Angular momentum***

Let us construct another conserved quantity M as

$$M^{\rho\mu\nu} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}\,. \tag{10}$$

(a) Show that

$$J^{\mu\nu} \equiv \int d^3x M^{0\mu\nu} \tag{11}$$

is antisymmetric and that it can be interpreted as the angular momentum of the system. *(Look at J^{ij})*

(b) How does $J^{\mu\nu}$ transform under $x^\mu \to x'^\mu = x^\mu + a^\mu$?
What is the physical interpretation of the extra terms?

(c) Show that the quantity $S_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}u^\sigma$ (where $u^\sigma = p^\sigma/\sqrt{-p\cdot p}$ is the system's four-velocity) is invariant under the translation a^μ. What are the components of S in the system's centre-of-mass frame? What is the physical interpretation of S?