Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
14. Apr. 2008
SS 08

Exercises on Elementary Particle Physics II

Prof. Dr. H.-P. Nilles

1. *Lie groups and Lie algebras - a first example*

   This excersise is designed to gain an intuitive understanding of Lie groups and algebras by generalizing well-known concepts encountered in explicit considerations.

   Consider the special unitary group $\mathrm{SU}(2) := \{g \in \mathrm{GL}(2,\mathbb{C}) | g^\dagger = g^{-1},\ \det g = 1\}$.

   (a) Show that $\mathrm{SU}(2) \cong S^3$. *Hint: Find an equation constraining the parameter space of g to S^3 as a submanifold in $\mathbb{R}^4$.*

   (b) Introduce spherical coordinates on S^3 to infer

   $$g(\omega,\theta,\phi) = \cos(\omega)\cdot e + i\,(\vec{\omega_0}\cdot\vec{\sigma})\sin(\omega), \quad \vec{\omega_0} \in S^2,$$

   with $\vec{\omega_0} = (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta))^T$. What is the range of the parameters $x^i := (\omega,\theta,\phi)$? What is the geometrical locus of the $\vec{\omega} := \omega\cdot\vec{\omega_0} \in \mathbb{R}^3$? The σ_i are the Pauli matrices defined by

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   They form a basis of the real vector space of hermitian traceless matrices.

   (c) Check that every $g \in \mathrm{SU}(2)$ obeys the differential equation

   $$\frac{\partial g}{\partial \omega} = i\,(\vec{\omega_0}\cdot\vec{\sigma})\,g.$$

   Integrate this to determine the solution

   $$g(\vec{\omega}) = e^{i\omega^i\sigma_i}$$

   and calculate $\frac{\partial g}{\partial \omega^i}\Big|_{\omega=\vec{0}}$.
   Hint: Do not forget to use the initial conditions.

Let us recapitulate our observations. First we showed that SU(2) is, besides its group properties, also a non-trivial geometrical object. Then, we exploided this fact by introducing coordinates on $\mathrm{SU}(2) \cong S^3$. Finally, we combined the realization of SU(2) as matrices with well-defined multiplication and a certain differential equation to find a (local) parametrization of any $g \in \mathrm{SU}(2)$ in terms of very local data, namely $\frac{\partial}{\partial \omega^i}\big|_0 g \in T_e(\mathrm{SU}(2))$. This is the Lie algebra $su(2)$ of SU(2). The whole program above relied just on the geometric structure of SU(2) and the combination with its algebraic properties as a group.

This is, what lies at the heart of the theory of Lie groups and Lie algebras, in general.

2. *From Lie groups to representations - glimpse with SU(2)*

   - A **Lie group** G is a group endowed with the structure of a differentiable manifold such that the operations

     (a) $\cdot : \; G \times G \;\to\; G\,, \quad (g,h) \;\mapsto\; g \cdot h$

     (b) $^{-1} : \; G \;\to\; G\,, \quad g \;\mapsto g^{-1}$

     are differential maps of differentiable manifolds.
     Using coordinates x^i on G we are able to define the basis $\frac{\partial}{\partial x^i}\big|_0 g =: T_i$ of the tangent space $\mathfrak{g} := T_e G$ at the identity $e \in G$. The T_i are called **generators of the Lie algebra** $\mathfrak{g}$ that has the following structure.

   - A Lie algebra $\mathfrak{g}$ is a vector space (from $T_e G$) together with a binary operation

     $$[\cdot,\cdot] \; : \; \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

     satisfying the conditions:

     (a) It is bilinear, i.e. linear in both entries. *(-Up to here, $\mathfrak{g}$ is an $\mathbb{R}$-algebra.-)*

     (b) It is skew-symmetric: $[a,b] = -[b,a]$ for $a, b \in \mathfrak{g}$

     (c) It fulfills the Jacoby identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ for $a, b, c \in \mathfrak{g}$

     It can be shown that $g = e^{ix^i T_i}$, at least locally. Hence, it makes sense to ananlyse (local) properties of G and its representations in terms of its corresponding Lie algebra $\mathfrak{g}$, which is a lot easier to deal with.

   - A **representation** ρ **of a Lie algebra** $\mathfrak{g}$ on a vector space V is a mapping

     $$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

     which is an algebra homomorphism, i.e. it is a homomorphism of vector spaces and fulfills $\rho([a,b]) = [\rho(a), \rho(b)]$, $\forall a, b \in \mathfrak{g}$.
     The dimension of V is called the **dimension of the representation** ρ: $\dim(\rho) := \dim(V)$.
     If there is a vector space $W \subset V$ so that $\rho(W) \subset W$, then the representation is called **reducible** and V is called the invariant subspace. If there are only $W = \{0\}$, V with this property then the representation ρ is called **irreducible**. In other words: a representation is irreducible, iff the only invariant subspace is V itself ($\{0\}$ is trivial).

Let us consider the example of $su(2)$ once again.

(a) Show in a slightly more abstract way that $su(2)$ is the set of all traceless hermitian matrices and prove that it is a Lie algebra.

Hint: $detA = exp\ Tr\ logA$ *and set* $[A, B] := A \cdot B - B \cdot A$.

(b) Perform a complex basis change from the Pauli matrices σ_i to

$$J_3 = \frac{1}{2}\sigma_3, \qquad J_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \qquad J_- = \frac{1}{2}(\sigma_1 - i\sigma_2),$$

and verify the commutation relations

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-, \qquad [J_+, J_-] = 2J_3 .$$

Next, we aim for classifying **all irreducible, finite-dimensional representations** ρ of $su(2)$ on a vector space V. Thus, it is important to note that $\rho(J_i)$, $i = 3, +, -$, are $n \times n$-matrices with $n := \dim(V)$ and $n \neq 0$ in general.

(c) Since J_3 is diagonal, $\rho(J_3)$ can also be chosen to be diagonal. Therefore V can be decomposed into eigenspaces of $\rho(J_3)$,

$$V = \bigoplus V_\alpha,$$

where α labels the eigenvalue of $\rho(J_3)$, i.e.

$$(\rho(J_3))v = \alpha v, \qquad v \in V_\alpha, \quad \alpha \in \mathbb{C}$$

(shorthand: write J_i for $\rho(J_i)$). Show that $J_+(v) \in V_{\alpha+1}$ and $J_-(v) \in V_{\alpha-1}$.

(d) Prove that all complex eigenvalues α which appear in the above decomposition differ from one another by 1.

Hint: Choose an arbitrary $\alpha_0 \in \mathbb{C}$ *from the decomposition and prove that*

$$\bigoplus_{k\in\mathbb{Z}} V_{\alpha_0+k} \subset V$$

is indeed equal to V *using the irreducibility of the representation.*

(e) Argue that there is $k \in \mathbb{N}$ for which $V_{\alpha_0+k} \neq 0$ and $V_{\alpha_0+k+1} = 0$. Define $n := \alpha_0 + k$. Note that up to now, we only know that $n \in \mathbb{C}$.

Draw a diagram. Write the vector spaces V_{n-2}, V_{n-1} and V_n in a row and indicate the action of J_3, J_+ and J_- on these vector spaces by arrows.

The eigenvalue n is called highest weight and a vector $v \in V_n$ is called highest weight vector. Is it clear why?

(f) Choose an arbitrary vector $v \in V_n$ (highest weight vector). Prove that the vectors $v, J_-v, J_-^2v, \dots$ span V.

Hint: Show that the vector space spanned by these vectors is invariant under the action of J_3, J_+ *and* J_- *and use the irreducibility of the representation.*

(g) Argue that all eigenspaces V_α are 1-dimensional.

(h) Prove that n is a non-negative integer or half-integer and that

$$V = V_{-n} \oplus \ldots \oplus V_n \ .$$

Complement your diagram drawn in part (e). What is the dimension of V?
Hint: The representation is finite dimensional, so there exists $m \in \mathbb{N}$ for which $J_-^{m-1} v \neq 0$ and $J_-^m v = 0$. Evaluate the product $J_+ J_-^m v$.

(i) Decompose the tensor product of a 2-dimensional and a 3-dimensional irreducible representation of $su(2)$,

$$V = V^{(2)} \otimes V^{(3)},$$

into two irreducible representations of dimension two and four: $\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{4}$.
Hint: For the definition of a tensor product of two representations see 3. (b). Note that the eigenvalue of J_3 on V is the sum of the eigenvalues of J_3 on $V^{(2)}$ and $V^{(3)}$. Draw the diagrams of the eigenvalues (with multiplicities). Then use the fact that the eigenspaces of J_3 on an irreducible representations are all 1-dimensional to show that V is reducible.

3. *Elementary constructions of representations*

Using the fact that the Lie algebra $\mathfrak{g}$ closes under $[\cdot, \cdot]$ we have an expansion

$$[T_i, T_j] = i f_{ijk} T_k, \qquad \forall\, T_i \in \mathfrak{g}, \tag{1}$$

where the coefficients f_{ijk} are called *structure constants* of $\mathfrak{g}$.[1]

(a) Determine the structure constants of $su(2)$ in the σ_i basis. Compare this with the algebra of the angular momentum operators (or $so(3)$).

As a representation ρ preserves the entire structure of the Lie algebra the representation matrices have to obey (1). Thus, this is an equivalent way to define a representation. Let us construct new representations out of old ones.

(b) Let $(\rho_1, V_1),\ (\rho_2, V_2)$ be two representations. Prove that $\rho_\oplus(T_i) := \rho_1(T_i) \oplus \rho_2(T_i)$, $\rho_\otimes(T_i) := \rho_1(T_i) \otimes \mathbb{1} + \mathbb{1} \otimes \rho_2(T_i)$ define represenations on $V_1 \oplus V_2$, $V_1 \otimes V_2$, respectively.

(c) Prove that $ad(T_i)_{kj} := i f_{ijk}$ defines a representation $(ad, \mathfrak{g})$ called the **adjoint representation of $\mathfrak{g}$**. Abstractly, this is given by $ad: X \mapsto [X, \cdot],\ \forall\, X \in \mathfrak{g}$.

(d) Show using equation (1) that $\bar{\rho}(T_i) := -\rho(T_i)^*$ defines a representation called the **complex conjugate representation of ρ**. ρ is said to be **real** if it is equivalent[2] to its complex conjugate $\bar{\rho}$.

[1] The i is necessary to guarantee a consistent expression for hermitian (representations of) generators T_i as needed for unitary representations. Strictly speaking, $[\cdot, \cdot]$ doesn't close anymore in $\mathfrak{g}$ as $[T_i, T_j]$ is antihermitian for hermitian T_i. This is the discrepancy between mathematicians and physicists.

[2] (ρ_1, V_1), (ρ_2, V_2) are called equivalent if there is a vector space isomorphism $\alpha : V_1 \to V_2$ obeying $\rho_1(X) = \alpha^{-1} \circ \rho_2(X) \circ \alpha,\ \forall\, X \in \mathfrak{g}$.