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Exercise 3
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SS 08

Exercises on Elementary Particle Physics II

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1. *Orthonormal root basis & fundamental weights of $SU(5)$ and $SO(10)$*

   We have seen in the example of Ex. 2.1 that it is possible to consider the roots α of $\mathfrak{g}$ also as r-dimensional vectors with $r = \dim(H)$. In order to deal only with (half) integer coordinates between -2 and 2 for the simple roots, it is convenient to embed the roots into higher dimensional vector spaces.
   For $su(n)$ this **orthonormal basis** is in $\mathbb{R}^n = \mathbb{R}^{r+1}$ and given by $((e_i)^j = \delta_i^j)$

$$\alpha > 0 : \quad e_i - e_j, \quad 1 \leq i < j \leq n, \qquad \alpha \text{ simple}: \quad e_i - e_{i+1}, \quad i = 1, \ldots, n-1. \tag{1}$$

   (a) Check that these roots define $su(n)$ by calculating A_{ij} using the standard scalar product in $\mathbb{R}^n$.

   (b) The Dynkin coefficients of the highest weight of the adjoint of $su(n)$ are $(1, 0^{n-3}, 1)$.[1] What is the corresponding **highest root** in (1)? Check (not explicitly) that the highest weight construction, cf. Ex. 2.2, produces all roots of $su(n)$, hence the adjoint. Do it explicitly for $su(3)$. *Hint: It is crucial to identify H corrrectly.*

   Introduce the **fundamental weights** μ^j by demanding Dynkin coefficients

$$\left(\Lambda^j\right)_i = 2\frac{\langle \mu^j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \stackrel{!}{=} \delta_i^j. \tag{2}$$

   For $\langle \alpha_i, \alpha_i \rangle = 2$ as in (1) the μ^j are just dual to the simple roots α_i. Each μ^j corresponds to an irreducible representation and a highest weight Λ of any representation can be expanded as $\Lambda = \sum_i \Lambda_i \mu^i$. Thus, the in general reducible representation of Λ can be constructed as a tensor product of the form $V_\Lambda = \bigotimes_{i=1}^r \left(V_{\mu^i}^{\otimes \Lambda_i}\right)$.

   (c) As a warm-up: What are the fundamental weights of $su(3)$ and the corresponding representations? Draw the weight vectors of the representation with highest weight given by the Dynkin coefficients $(0, 1)$ in the two-dimensional picture of Ex. 2.2, (c). What is the relation to the representation of $(1, 0)$?
   Hint: What are the generators H_{α_i} of Ex. 2.1, (h), for the scalar product in (a)?

[1] 0^k means k zeros in a row.

(d) Determine the four fundamental weights of $su(5)$ and construct the entire representations, denoted by $\mathbf{5}, \mathbf{10}, \mathbf{10'}, \mathbf{5'}$, by the highest weight construction. How can you relate two representations, respectively?

Analogous to $su(n)$, one can analyse $so(2n)$. It is the Lie algebra of $2n \times 2n$ antisymmetric matrices and has rank n. It has the following roots:

$$\alpha > 0: \quad e_i \pm e_j, \quad 1 \leq i < j \leq n, \quad \alpha \text{ simple}: \quad e_i - e_{i+1}\,,\ i = 1, \ldots, n-1;\ e_{n-1} + e_n. \tag{3}$$

(e) Calculate the Cartan matrix A of $so(2n)$ and draw the Dynkin diagram.

(f) Specialize to $so(10)$ and determine the five fundamental weights. Construct all weights and the dimensions of the representations $\mathbf{10}$, $\mathbf{16'}$ and $\mathbf{16}$ corresponding to the Dynkin coefficients $(1, 0^4)$, $(0^3, 1, 0)$ and $(0^4, 1)$, respectively. How are they related?
(Optional !) What are the weights of $(0, 1, 0^3)$?

Elementary particles are supposed to sit in the fundamental representations of the gauge group. Hence, the general idea of <u>G</u>rand <u>U</u>nified <u>T</u>heories is to enhance the gauge group such that all fundamental particles (of one familiy) can be organized in a fundamental representation of the GUT gauge group.

2. *Group theoretical symmetry breaking of SU(5) and SO(10) GUTs*

The underlying GUT gauge group is only visible at high energies about the GUT-scale. Thus, it has to be broken to the gauge group of the <u>S</u>tandard <u>M</u>odel at low energies. Representations of the larger GUT group break into those of the SM gauge group. Hence, tools for this group theoretical symmetry breaking have to be applied.

Dynkin's Symmetry Breaking:

To each simple root one assigns an integer number, called the **Kac-label** a_i. They are given as the coefficients of the decomposition of the highest root in the basis of simple roots. Deleting any node with Kac-label $a_i = 1$ from the Dynkin diagram gives a maximal regular subalgebra times a U(1) factor.

(a) In the case of SU(5), all Kac-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the Standard model gauge group, i.e.

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

The U(1) generator is constructed as a sum of the roots of SU(5) such that it is orthogonal to all roots of SU(3) × SU(2). Show that $Q = (\text{-}2, \text{-}2, \text{-}2, 3, 3)$ fulfills these conditions.

(b) The **5** of SU(5) of Ex. 3.1, (d), is a reducible representation of the SU(3) × SU(2) × U(1) subgroup. Let α_1 and α_2 correspond to SU(3) and α_4 to SU(2). Thus, every weight λ of SU(5) decomposes as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \rightarrow (\lambda_1, \lambda_2 | \lambda_4) = (\mu|\nu).$$

First, write down all weights $(\mu|\nu)$, then find the highest weight μ and determine all weights and the dimension of the corresponding representation. Consider now the values of ν belonging to this μ-representation and state the dimension of the ν-representation! Repeat these steps starting with the highest weight ν. Finally, determine the U(1) charge by applying the U(1) generator to the weight vectors. The result reads

$$\mathbf{5} \rightarrow (\mathbf{3}, \mathbf{1})_{\text{(-2)}} \oplus (\mathbf{1}, \mathbf{2})_{\text{(3)}}.$$

(c) Repeat the analysis for the representation **10** and verify

$$\mathbf{10} \rightarrow (\mathbf{1}, \mathbf{1})_{\text{(6)}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{\text{(-4)}} \oplus (\mathbf{3}, \mathbf{2})_{\text{(1)}}.$$

Hint: All weights which appear in the calculation have multiplicity 1.

(d) Perform the breaking for the representation corresponding to the highest weight with Dynkin coefficients $(1, 0, 0, 1)$, i. e. the adjoint **24**. The result reads

$$\mathbf{24} \rightarrow (\mathbf{8}, \mathbf{1})_{\text{(0)}} \oplus (\mathbf{1}, \mathbf{3})_{\text{(0)}} \oplus (\mathbf{1}, \mathbf{1})_{\text{(0)}} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{\text{(5)}} \oplus (\mathbf{3}, \mathbf{2})_{\text{(-5)}}.$$

Identify the gauge group of the standard model.
Hint: All weights which appear in the calculation have multiplicity 1, except for $(0, 0, 0, 0)$ *in* **24** *of* $SU(5)$ *with multiplicity 4 and* $(0, 0)$ *in* **8** *of* $SU(3)$ *with multiplicity 2. What is the origin of this?*

After a renormalization of the U(1) generator to $Q' = \frac{1}{6}Q$ we recover one family of the standard model in $\bar{\mathbf{5}} \oplus \mathbf{10}$. However, we have not achieved a complete unification into only one fundamental representation of the GUT group. Therefore, we increase the gauge group once more to SO(10).

(e) Decompose **16** into irreducible representations of SU(5) × U(1) by deleting an appropriate root in the Dynkin diagram of SO(10). This is a complete unification of one family predicting a right-handed neutrino, too:

$$\mathbf{16} \rightarrow \mathbf{1}_{\text{(-5)}} \oplus \bar{\mathbf{5}}_{\text{(3)}} \oplus \mathbf{10}_{\text{(-1)}}.$$

(f) The SM-Higgs is contained in **10**. Decompose it as well:

$$\mathbf{10} \rightarrow \mathbf{5}_{\text{(2)}} \oplus \bar{\mathbf{5}}_{\text{(-2)}}.$$