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Exercise 5
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Exercises on Elementary Particle Physics II

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1. *The Lorentz group, $SL(2,\mathbb{C})$ and Weyl spinors*

Let us recall some basic facts about the Lorentz group $\mathrm{SO}(1,3)$ and its representations. The Lie algebra $so(1,3)$ is defined by $\lambda^T = -\eta\lambda\eta$ for $\lambda \in so(1,3)$ and a convenient basis is given by $(M^{\mu\nu})^{\rho}_{\ \sigma} = i\,(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma})$. Considering $so(1,3)\otimes\mathbb{C}$ allows the complex basis change

$$T^i_{L/R} := \frac{1}{2}(J^i \pm iK^i) \quad \text{for} \quad J^i := \frac{1}{2}\epsilon^{ijk}M^{jk},\ K^i := M^{0i} \tag{1}$$

such that the algebra decouples into $su(2)\oplus su(2)$,

$$\left[T^i_{L/R}, T^j_{L/R}\right] = i\epsilon^{ijk}T^k_{L/R}, \qquad \left[T^i_L, T^j_R\right] = 0. \tag{2}$$

Moreover, this is precisely the algbra of $sl(2,\mathbb{C})$ and we obtain the result $so(1,3) \cong sl(2,\mathbb{C})\otimes\mathbb{C}$. Thus, every representation of $so(1,3)$ can be characterized by the spins of the two $su(2)$'s, namely a pair (j_1, j_2) with $j_i \in \mathbb{N}_0/2$.

However, the Lorentz group $\mathrm{SO}(1,3)$ is not equal to $\mathrm{SL}(2,\mathbb{C})$ as there are topological differences that go beyond the equivalence of the algebras. Let us establish this connection from the viewpoint of $\mathrm{SL}(2,\mathbb{C})$.

(a) Consider the map from $\mathbb{R}^4$ to the hermitian 2×2-matrices defined by $x^\mu \mapsto X = x^\mu\sigma_\mu$ with $\sigma^\mu = (\mathbb{1}, \sigma_i)$. Show that $\det(X) = x^\mu x_\mu$ and argue that $y^\mu = \Lambda^\mu_{\ \nu}x^\nu$ for $\Lambda \in \mathrm{SO}(1,3)$ induces a map $AXA^\dagger$ for $A \in \mathrm{SL}(2,\mathbb{C})$, i.e. $\det(A) = 1$. Reverse the argument: Each $A \in \mathrm{SL}(2,\mathbb{C})$ gives rise to a Lorentz transformation $\Lambda(A)$.

(b) Check that $\Lambda(\cdot)$ defines a representation of $\mathrm{SL}(2,\mathbb{C})$ and determine its kernel, i.e. $\mathrm{Ker} := \{A \in \mathrm{SL}(2,\mathbb{C}) \mid \Lambda(A) = \mathbb{1}\}$. Use this to show that $\Lambda(-A) = \Lambda(A)$. *Hint: Specialize to $x^\mu = (1,\vec{0})$. Then, use Schur's Lemma, i.e the fact that for a hermitian A with $[A, X] = 0\ \forall X$ hermitian 2×2-matrices follows $A = c\mathbb{1}$.*

Thus $\mathrm{SO}(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, i.e. $\mathrm{SL}(2,\mathbb{C})$ is its simply connected double cover.[1] Furthermore, a four-vector is a representation of $\mathrm{SL}(2,\mathbb{C})$ as well as $\mathrm{SO}(1,3)$. However, a representation of $\mathrm{SL}(2,\mathbb{C})$ lifts only to a representation of

[1] Strictly speaking, $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ is only isomorphic to one of the four connected components of $\mathrm{SO}(1,3)$.

SO(1, 3) if $\pm\text{id} \in \text{SL}(2,\mathbb{C})$ is represented by $\mathbb{1}$. The group $\text{SL}(2,\mathbb{C})$ exhibits so-called **spinors** for that this is not fulfilled. These are just its fundamental representations $(1/2, 0)$ and $(0, 1/2)$ defined by

$$\psi_\alpha \mapsto \psi'_\alpha = M_\alpha{}^\beta \psi_\beta, \qquad \bar{\psi}_{\dot{\alpha}} \mapsto \bar{\psi}'_{\dot{\beta}} = M^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}, \qquad M \in \text{SL}(2,\mathbb{C}) \tag{3}$$

The undotted and dotted spinors are the **left-** and **right-chiral Weyl spinors**. Two-dimensional representation matrices obeying eqn. (2) are just the Pauli matrices (and unitarily equivalent matrices), thus one can use the exponential map to write

$$D_L := M = \exp\left((a_i + ib_i)\sigma_i\right), \qquad D_R := M^* = \exp\left((a_i - ib_i)\sigma_i^*\right). \tag{4}$$

(c) Prove using $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$ the identities

$$D_L^\dagger = \sigma_2 D_R^{-1}\sigma_2, \qquad D_L^T \sigma_2 D_L = \sigma_2 \tag{5}$$

and argue that σ_2 is a spinor metric. How transforms $(\psi_\alpha)^*$? Set $(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}}$. Use (a) to determine the spinor indices of σ^μ.

(d) One can use σ_2 to raise and lower spinor indices. Introduce $\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma_2$. What are the inverse tensors denoted by $\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}$? Determine the transformation behavior of $\psi^\alpha := \epsilon^{\alpha\beta}\psi_\beta$ and $\bar{\psi}^{\dot{\alpha}} := \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$.

(e) Show that $\psi\phi := \psi^\alpha\phi_\alpha$, $\bar{\psi}\bar{\phi} := \bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}}$ as well as $(\psi_\beta)^* \,\epsilon^{\dot{\beta}\dot{\alpha}}\bar{\phi}_{\dot{\alpha}} = i\psi^\dagger\sigma_2\bar{\phi}$, $\left(\bar{\psi}_{\dot{\beta}}\right)^* \epsilon^{\beta\alpha}\phi_\alpha$ are Lorentz scalars. How does $\psi\sigma^\mu\bar{\phi} := \psi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\phi}^{\dot{\beta}}$ transform?

(f) Introduce $\bar{\sigma}^\mu := (\mathbb{1}, -\sigma_i)$ and check $\bar{\sigma}^\mu = i\sigma_2(\sigma^\mu)^*(-i\sigma_2)$, thus $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (\sigma^\mu)^{\dot{\alpha}\beta}$. Check also $(\bar{\sigma}^\mu)^T = i\sigma_2\sigma^\mu(-i\sigma_2)$ and $(\sigma^\mu)^{\alpha\dot{\beta}} = (\bar{\sigma}^\mu)^{\dot{\beta}\alpha}$. Finally, determine the index structure of the spin generators

$$\sigma^{\mu\nu} := \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right), \quad \bar{\sigma}^{\mu\nu} := \frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right). \tag{6}$$

The spin generators furnish a representation of the Lorentz algebra such that

$$D_L = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \qquad D_R = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right) \tag{7}$$

for eqn. (4) with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and $\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) \in \text{SO}(1,3)$.

2. *Weyl Spinors are Grassmann valued*

In Ex. 5.1 we have defined an inner product for Weyl spinors. In the following we consider left-chiral Weyl spinors ψ_α, ϕ_α and θ_α.

(a) Consider the pairing $\psi\psi$. Which relation do the ψ_α have to fulfill in order for this to be non-vanishing? Assume this for all Weyl spinors to show that $\psi^\alpha\phi_\alpha = -\phi_\alpha\psi^\alpha$, $\bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = -\bar{\phi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}$ and $\psi\phi = \phi\psi$ as well as $\bar{\psi}\bar{\phi} = \bar{\phi}\bar{\psi}$.

(b) Prove the relations

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \qquad \theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta,$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta, \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta,$$

$$(\theta\phi)(\theta\psi) = -\tfrac{1}{2}(\phi\psi)(\theta\theta), \qquad (\bar\theta\bar\phi)(\bar\theta\bar\psi) = -\tfrac{1}{2}(\bar\phi\bar\psi)(\bar\theta\bar\theta).$$

(c) Check also

$$(\bar\phi\bar\sigma^\mu\psi) = -(\psi\sigma^\mu\bar\phi), \qquad (\phi\sigma^\mu\bar\psi)^* = (\bar\phi\bar\sigma^\mu\psi),$$
$$\psi_\alpha\bar\phi_{\dot\beta} = \frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}(\psi\sigma_\mu\bar\phi), \qquad (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta).$$

In summary, the components of spinors are Grassmann variables, i.e. anti-commuting. It is also possible to introduce differentiation w.r.t. a Grassmann variable θ_α by differential operators

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad \text{and} \quad \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}}$$

obeying $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$ and $\bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha}$. However, the Leibniz rule includes a minus sign for consistency,

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma.$$

(d) Show that $\partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta$.

(e) Check that

$$\partial^\alpha\partial_\alpha(\theta\theta) = \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 4.$$

Identical to differentiation one introduces integration by

$$\int d\theta^\alpha = 0, \qquad \int d\theta^\alpha\theta_\alpha = 1 \quad \text{(no summation)} \tag{8}$$

that is linear and automatically defined on arbitrary functions $f(\theta)$. The volume elements are defined by

$$d^2\theta := -\tfrac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}, \qquad d^2\bar\theta := -\tfrac{1}{4}d\bar\theta_{\dot\alpha}d\bar\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta},$$
$$d^4\theta := d^2\theta d^2\bar\theta. \tag{9}$$

This implies

$$\int d^2\theta\,(\theta\theta) = \int d^2\bar\theta\,(\bar\theta\bar\theta) = 1, \tag{10}$$

which can be checked analogously to differentiation. Note that integration just projects on the highest θ or $\bar\theta$ component in the finite Taylor expansion of a function $f(\theta,\bar\theta) = c_{(0,0)} + \ldots + c_{(2,0)}\theta^2 + c_{(0,2)}\bar\theta^2 + \ldots + c_{(2,2)}\theta^2\bar\theta^2$, $c_{(i,j)} \in \mathbb{C}$.