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Exercise 2
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SS 09

Exercises on Theoretical Astroparticle Physics

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In this exercise sheet we want to take a look at the Lie algebras $\mathfrak{su}(N)$ and their representation. However, we will quickly specialize to the $\mathfrak{su}(5)$ case. We remember that $\mathfrak{su}(N)$ is the space of traceless hermitian complex $N \times N$ matrices.

1. (a) Give a real basis of $\mathfrak{su}(N)$ in terms of the elementary $N \times N$ matrices $(e_{ij})_{kl} := \delta_{ik}\delta_{jl}$ analogously to the Pauli matrices. What is the dimension of $\mathfrak{su}(N)$?

   (b) Next we define the **Cartan algebra** $\mathfrak{h}$ as a maximal set of commuting matrices in $\mathfrak{g} = \mathfrak{su}(N)$

   $$[X, Y] = 0 \quad \forall X, Y \in \mathfrak{h}.$$

   It is not hard to see that they form a subspace of $\mathfrak{su}(N)$. We know that such a set can be diagonalized simultaneously and so we can choose $\mathfrak{h}$ to be the set of diagonal matrices in $\mathfrak{su}(N)$. The dimension of the Cartan algebra is called **rank** of the Lie algebra $r := \dim\mathfrak{h}$. Find a basis of $\mathfrak{h}$ for $\mathfrak{su}(N)$. What is the rank of $\mathfrak{su}(N)$? You should find

   $$\mathfrak{h} = \left\{ H_\lambda := \sum_i \lambda_i e_{ii} \Big| \sum_i \lambda_i = 0 \right\}$$

   (c) Now we also want to diagonalize the action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, which is given by the commutator. Perform a complex base change (cf. 1st sheet: $J_1, J_2 \to J_+, J_-$) to an eigenbasis of all elements H_λ of $\mathfrak{h}$. You should find e_{ij} as the eigenbasis and

   $$\mathrm{Ad}(H_\lambda)e_{ij} = [H_\lambda, e_{ij}] = (\lambda_i - \lambda_j)e_{ij}\,.$$

This equation can also be seen as a map

$$\begin{aligned}\alpha_\bullet : \mathfrak{g}/\mathfrak{h} &\to \mathfrak{h}^* \\ e_{ij} &\to \alpha_{e_{ij}} : (H_\lambda \to (\lambda_i - \lambda_j))\,,\end{aligned}$$

where $\mathfrak{h}^*$ denotes the dual space of $\mathfrak{h}$. The points $\alpha_{e_{ij}} \in \mathfrak{h}^*$ are called **roots** and they span the **root lattice** which is isomorphic to $\mathbb{Z}^r$. A basis of $\mathfrak{h}$ is e.g. given by the set

$$\left\{ H_i = e_{ii} - e_{N,N} \big| i = 1 \cdots r = N - 1 \right\}\,.$$

Using the dual basis

$$\{H_i^*\} \qquad \text{with} \;\; H_i^*(H_j) = \delta_i j\,,$$

we can expand each root in this basis

$$\alpha = c_i H_i^* \qquad \text{where} \;\; c_i = \alpha(H_i)\,.$$

Now we can define

- a **positive root** as a root $\alpha > 0$, whose first non-vanishing expansion coefitient c_i is positive.
- a **simple root** as a positive root α, which cannot be written as a sum of two other positive roots α', α''.

(d) Show that the positive roots of $\mathfrak{su}(N)$ are the $\alpha_{e_{ij}}$ with $1 \leq i < j \leq N$ and the simple roots are the $\alpha_i := \alpha_{e_{i,i+1}}$ with $i = 1 \cdots r$. Is the set of simple roots a basis of $\mathfrak{h}^*$?

(e) There is a natural scalar product on $\mathfrak{h}^*$, which can be computed as follows: For $H_\lambda = \lambda_i e_{ii}$ we can write $\alpha(H_\lambda) = \alpha^i \lambda_i$ with the additional constraint that $\sum_i \alpha^i = 0$. Now we define:

$$\langle \alpha, \alpha' \rangle := \sum_i \alpha^i \alpha'^i$$

The **Cartan matrix** is defined as

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle},$$

with a set α_i of simple roots. Compute the Cartan matrix of $\mathfrak{su}(5)$. The Cartan matrix can be visualized nicely by the **Dynkin diagram**. It consists of a small circle for every simple root α_i and $A_{ij}A_{ji}$ lines joining the circles α_i and α_j for all i, j. Draw the Dynkin diagram of $\mathfrak{su}(5)$.

Although the choice of the basis of the root space $\mathfrak{h}^*$ is arbitrary (and therefore the definition of positive roots etc. pp.), the Cartan matrix and the Dynkin diagram are unique for each Lie algebra $\mathfrak{g}$ up to permutations of the simple roots. One finds, that all informations on the Lie algebra (i.e. the structure constats f_{ijk}) are encoded in the Dynkin diagram. Hence, we can classify all simple Lie algebras by their Dynkin diagrams. The Lie algebras $\mathfrak{su}(N)$ are also called A_{n-1}, A stands for this type of Dynkin diagram and $n-1$ for the rank.

2. The next task is to construct all irreps of $\mathfrak{su}(N)$. This works pretty similar to the case of $\mathfrak{su}(2)$, just in a bigger setup. Instead of J_3 we now diagonalize the cartan subalgebra $\mathfrak{h}$ with some basis $\{H_i\}$. Hence the representation space V decomposes into simultaneous eigenspaces of the H_i

$$V = \bigoplus_\mu V_\mu \qquad \text{with} \quad H_i v = \mu(H_i) v \quad \text{for} \quad v \in V_\mu\,.$$

The map $\mu \in \mathfrak{h}^*$ is called the **weight** of the state v. Note that again we write H_i for $\rho(H_i)$. The generalization of the raising and loweing operators $J_\pm$ is the space $\mathfrak{g}/\mathfrak{h}$, whose basis element, corresponding to the root α we will call E_α (e.g. $\alpha = \alpha_{e_{ij}}$). Recall that

$$[H_i, E_\alpha] = \alpha(H_i) E_\alpha\,.$$

For $\alpha > 0$ we will call the E_α raising and $E_{-\alpha}$ lowering operator. Note that for a given α they always appear together.

(a) Show that for $v \in V_\mu$ we get $E_\alpha v \in V_{\mu+\alpha}$.

(b) What are the weights of the adjoint representation?

(c) Take the example $\mathfrak{su}(5)$ and the fundamental representation **5** with the basis of the Cartan algebra as given above. What are the eigenspaces V_μ and the corresponding weights μ?

(d) For a representation ρ with weights $\mu^i, i = 0 \cdots \dim \rho$, what are the weights of the conjugate representation $\bar{\rho}$?

Now in principle we are ready to construct the representations. For a finite dimensional representation we will find a state with **highest weight** Λ, which is annihilated by all positive root operators. Then we can get all states by acting with the lowering operators on it. In order to do this, we represent the weights by the Dynkin lables. For a weight μ we define the **Dynkin label**

$$m_i := \frac{2\langle \mu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} .$$

The Dynkin labels always consist of integer numbers which for a highest weight state are non negative. It is easy to see, that acting with $E_{-\alpha_i}$ corresponds to subtracting the ith row of the Cartan matrix from the Dynkin label. Now you can construct all irreducible representations via the following recipe.

- Start with the Dynkin label m with non negative entries, representing the highest weight state.
- If the ith entry of the Dynkin label m_i is positive, you can get m_i new states by subtracting m_i times the ith row of the Cartan matrix.
- Repeat the last step for all new states, for $i = 1 \cdots r$.
- At the end you should arrive at the lowest weight state with only non positive entries in the Dynkin label.

Every Dynkin label m which you get this way corresponds to a weight μ and therefore to a eigenspace V_μ of $\mathfrak{h}$. In most cases these eigenspaces are one dimensional so you can find the dimension of the representation by counting the Dynkin labels. One big exception is the adjoint representation, where the Dynkin label $(0, \cdots, 0)$ is occupied r times.

(e) Construct the **5** and the **10** of $\mathfrak{su}(5)$ with the highest Dynkin labels $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. What are the highest Dynkin labels of the $\bar{\mathbf{5}}$ and the $\overline{\mathbf{10}}$? Also construct the adjoint, the **24**, from the Dynkin label $(1, 0, 0, 1)$. How can you see, that it is real?

3. Now we want to break the SU(5) GUT to the standard model SU(3)×SU(2)×U(1). We first decompose the Cartan algebra:

$$\mathfrak{h} = \mathfrak{h}_3 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_Y$$

where $\mathfrak{h}_3$ is spanned by H_1 and H_2, $\mathfrak{h}_2$ by H_3 and $\mathfrak{h}_Y$ is orthogonal on both of them.

(a) Show that the Hypercharge generator (or some multiple of it) is

$$Y = -2H_1 - 4H_2 - 6H_3 - 3H_4\,.$$

This decomposition also applies to the Dynkin labels of the states in a representation.

$$(m_1, m_2, m_3, m_4) \to (m_1, m_2 | m_4)$$

(b) Take the $\bar{\mathbf{5}}$, the $\mathbf{10}$ and the $\mathbf{24}$ and decompose them as follows. Compute also the Hypercharge of the states. You see, that one gets the matter content of one family of the SM.

$$\begin{aligned}
\bar{\mathbf{5}} &\to (\bar{\mathbf{3}}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2})_3 \\
\mathbf{10} &\to (\mathbf{3}, \mathbf{2})_1 \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_6 \\
\mathbf{24} &\to (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{2})_{-5} \oplus (\bar{\mathbf{3}}, \mathbf{2})_5
\end{aligned}$$