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Theoretische Physik

Exercise 4
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SS 09

Exercises on Theoretical Astroparticle Physics

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In this exercise sheet we will treat the seesaw mechanism and neutrino oscillations.

1. **Dirac-Majorana mass term and the seesaw mechanism**

   The most general mass term for for a Dirac spinor Ψ is called the *Dirac-Majorana mass term.* It is a combination of the Dirac and Majorana mass terms from the previous sheet.

   (a) Show
   $$(\Psi^c)_{L,R} = (\Psi_{R,L})^c \text{ and } \overline{(\Psi_{L,R})^c}(\Psi_{R,L})^c = \overline{\Psi_{R,L}}\Psi_{L,R}. \tag{1}$$

   (b) Using (a) show that
   $$\mathcal{L}_m = -\frac{1}{2}\left[2m_D\overline{\Psi_L}\Psi_R + m_L\overline{\Psi_L}(\Psi^c)_R + m_R\overline{(\Psi^c)_L}\Psi_R\right] \tag{2}$$
   is the same as the following
   $$-\frac{1}{2}\begin{pmatrix}\overline{\Psi_L} & \overline{(\Psi^c)_L}\end{pmatrix}\mathcal{M}\begin{pmatrix}(\Psi^c)_R \\ \Psi_R\end{pmatrix} \tag{3}$$
   where
   $$\mathcal{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R\end{pmatrix} \tag{4}$$
   is referred to as the *neutrino mass matrix.*

   (c) From now on we set $m_L = 0$ and $m_D \ll m_R$. We can diagonalize $\mathcal{M}$ using an orthogonal matrix A
   $$A^T\mathcal{M}A = \text{diag}(m_1, m_2). \tag{5}$$
   Show that to the first non-vanishing order in the (small) parameter $\rho := m_D/m_R$ that
   $$\begin{pmatrix} -m_D^2/m_R & 0 \\ 0 & m_R\end{pmatrix}. \tag{6}$$
   Find the rotation matrix A to the first order in ρ for the diagonalization.

   (d) Why do we have a problem with (6)? To circumvent this problem we write
   $$\begin{pmatrix} -m_D^2/m_R & 0 \\ 0 & m_R\end{pmatrix} = \begin{pmatrix} m_D^2/m_R & 0 \\ 0 & m_R\end{pmatrix}K^2, \tag{7}$$
   i.e. $\mathcal{M} = A^T\,\text{diag}(m_1, m_2)K^2A$. Determine the matrix K and write the $\mathcal{L}_m$ in mass eigenstates. Are these mass eigenstates Dirac particles or Majorana particles?

(e) We have assumed $\rho \ll 1$. What does this imply for mass eigenstates?

2. **Neutrino oscillation**

   Let's assume that there are n orthonormal flavour eigenstates $|\nu_\alpha\rangle$. These states are connected to the n mass eigenstates via an unitary mixing matrix U

$$|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle \tag{8}$$

   (a) Assuming that the mass eigenstate $|\nu_i\rangle$ are stationary states and were emitted with momentum p by a source at $x = 0$ and $t = 0$ what is the form of $|\nu_i(x,t)\rangle$?

   (b) What is the form of the Hamiltonian of deep relativistic particle?

   (c) Using (a) and (b) write down the time-dependent transition amplitude $A(\alpha \to \beta)(t)$ for a flavour convertion $\nu_\alpha \to \nu_\beta$. Simplify it further using $L = x = ct$ being the distance between source and detector and using $E \cong p$ for the neutrino energy.

   (d) Obtain the transition probability P from A. What is the probability of finding the original flavour?

   (e) We now assume that we have two flavours and have one mixing angle. What's the form of the unitary matrix U? Compute $P(\alpha \to \beta)$ for this case.