
Exercises on Group Theory

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–CLASS EXERCISES–

C 1.1 Properties of Groups

- (a) Show that for a given group element the inverse element is unique.
- (b) Show the “Rearrangement Theorem”, i.e. that $gG := \{gg' | g' \in G\} = G$.
- (c) Consider the Euclidean group $E(n)$, i.e. the group of transformation $\vec{x} \rightarrow O\vec{x} + \vec{b}$, $O \in O(n)$, $\vec{b} \in \mathbb{R}^n$. How does the identity element look like? What is the multiplication law? How does the inverse of (O, \vec{b}) look like?
- (d) What are the dimensions of the groups $O(n)$ and $U(n)$?
- (e) Find a formula for the inverse of a 2×2 matrix.
- (f) Show that every element of $SO(2)$ can be given in the form,

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.$$

- (g) Show that every element of $SU(2)$ can be given in the form,

$$\begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

with $|\alpha|^2 + |\beta|^2 = 1$.

- (h) Find all groups of order 4. Give their multiplication tables.

C 1.2 Conjugacy Classes and Normal Subgroups

- (a) Show that the conjugation $g \sim g'gg'^{-1}$ is an equivalence relation, i.e. that

- $g \sim g$

- $g \sim h \Rightarrow h \sim g$
- $g \sim h, h \sim k \Rightarrow g \sim k.$

A subgroup $H \subset G$ is called *normal* if it is self-conjugate, i.e. $gHg^{-1} = H \forall g \in G$. Show that:

- (b) A normal subgroup is a union of conjugacy classes.
- (c) All subgroups of Abelian groups are normal.
- (d) The center $\{g \in G | gh = hg \forall h \in G\}$ is always a normal subgroup.
- (e) A subgroup which contains half of the elements, $2|H| = |G|$, is always normal.
- (f) Find the order, elements and multiplication table for D_3 , the symmetry group of an equilateral triangle. What are the orders of the elements? Find the conjugacy classes. What are its subgroups? Which of them are normal?