Physikalisches Institut
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Theoretische Physik

Exercise 9
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SS 10

# Exercises on Group Theory

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## –Home Exercises–

### H 9.1 Matrix Identities

(a) Prove the following matrix identities:

- $(AB)^T = B^T A^T$
- $\operatorname{tr}[A, B] = 0$
- $\left(e^A\right)^T = e^{A^T} \qquad \left(e^A\right)^\dagger = e^{A^\dagger}$
- $e^{UAU^{-1}} = U e^A U^{-1}$
- If $\lambda$ is an eigenvalue of $A$ then $e^\lambda$ is an eigenvalue of $e^A$.
- $\det e^A = e^{\operatorname{tr} A}$
  *Hint: Bring $A$ to Jacobi form, $UAU^{-1} = J$, and write $J$ as a sum of a diagonal and a nilpotent matrix which commute. What is the exponential of a diagonal and of a nilpotent matrix?*

(b) Show the *Baker–Campbell–Hausdorff* formula to second order,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\mathcal{O}((A,B)^3)} .$$

### H 9.2 Subalgebras

Consider a Lie group $G$ with Lie algebra $\mathfrak{g}$ and a subspace $\mathfrak{h} \subset \mathfrak{g}$. Show:

(a) If $\mathfrak{h}$ is a closed subalgebra, i.e. $h_1, h_2 \in \mathfrak{h} \;\Rightarrow\; [h_1, h_2] \in \mathfrak{h}$, then $H = e^{\mathfrak{h}}$ is a subgroup of $G$.

(b) If $\mathfrak{h}$ is an invariant subalgebra, i.e. $h \in \mathfrak{h}$, $g \in \mathfrak{g} \;\Rightarrow\; [h, g] \in \mathfrak{h}$, then $H$ is a normal subgroup in $G$.

(c) If $\mathfrak{h}$ is a null space, i.e. $h \in \mathfrak{h}$, $g \in \mathfrak{g} \;\Rightarrow\; [h, g] = 0$, then $H$ is in the center of $G$.

### H 9.3 Algebraic equivalence of $SO(3)$ and $SU(2)$, part 2

(a) Prove the formula

$$e^{\mathrm{i}\vec{m}\cdot\vec{\sigma}} = \mathbb{1}\cos(m) + \mathrm{i}\sin(m)\ \hat{m}\cdot\vec{\sigma}$$

with $m = |\vec{m}|$, $\hat{m} = \vec{m}/m$. ($\sigma_i$: Pauli matrices, $m_i \in \mathbb{R}$)

(b) We write

$$SU(2) \ni U = e^{\mathrm{i}\varphi\hat{n}\cdot\sigma/2}$$

with $|\hat{n}| = 1$. Choosing $\hat{n}$ to be in the whole unit sphere, what is the parameter space of $\varphi$? What is the identification at the boundary?

(c) For $O \in SO(3)$ we have $O = e^{\alpha\hat{n}\cdot\vec{L}}$ with $0 \leq \alpha \leq \pi$ and $\hat{n}$ again in the unit sphere $S^2$ (see last sheet). Show that the map $\mu : (\varphi, \hat{n}) \mapsto (\alpha = \varphi \mod \pi, \hat{n})$ is a group homomorphism from $SU(2)$ to $SO(3)$. What is the group element associated to $\mu(\varphi = \pi, \hat{n})$? What is the preimage of $(\alpha, \hat{n})$ in terms of $SU(2)$ elements?

Since each $O \in SO(3)$ has exactly two preimages, we find that $SO(3) \cong SU(2)/\mathbb{Z}_2$ with $\mathbb{Z}_2 = \{\pm\mathbb{1}_2\}$. This fits nicely with the geometrical picture since the three-dimensional ball with opposite points at the boundary identified can be viewed as a three-sphere with opposite points identified. This space is also called *real projective space*, $\mathbb{PR}^3 = S^3/\mathbb{Z}_2$.

### H 9.4 Negative definite Killing form

Show that if the Killing form on a matrix Lie algebra $\mathfrak{g}$ is negative definite, i.e. $\langle X, X\rangle < 0$ for all $0 \neq X \in \mathfrak{g}$, then the Lie group $G = e^{\mathfrak{g}}$ is bounded.

### H 9.5 $U(n)$ decomposition

Remember that the Lie algebra of $U(n)$ consists of the Hermitean $n \times n$ matrices. Find a one-dimensional null space $\mathfrak{h} \subset \mathfrak{su}(n)$. Identify the associated subgroup of $U(n)$. This shows that $U(n)$ is not semi-simple.

### H 9.6 Adjoint representation

Consider a Lie algebra $\mathfrak{g}$ with basis $T_i$ and structure constants $[T_i, T_j] = f_{ijk}T_k$. Show that the *adjoint representation*, defined by

$$\mathrm{ad}(T_i)_{jk} = f_{ijk}$$

is a representation. What is its dimension?