Physikalisches Institut
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Theoretische Physik

Exercise 10
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 10.1 On Lie algebras and Killing froms

Let $\mathfrak{g}$ be a Lie algebra with basis $\{T_i\}$ and $g_{ij} = (T_i, T_j)$ a matrix representation of the Killing form.

(a) Let $X \in \mathfrak{Z}(\mathfrak{g})$. What is the matrix form of $\mathrm{ad}(X)$?

(b) Show: If $\mathfrak{g}$ contains an Abelian ideal, then g_{ij} is degenerate. *Hint: Choose a Basis such that $T_1, \ldots, T_m$ generate the Abelian ideal. Write g_{ij} in terms of the structure constants and the structure constants in terms of commutators. Show that $g_{1i} = 0$ for all i.*

(c) *Bonus:* Show that the converse is also true.

(d) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i$ simple. Show that the Killing form is block-diagonal.

(e) Let $\mathfrak{g}$ be semisimple. Show that every generator can be written as a sum of commutators.

H 10.2 $\mathfrak{su}(2)$ representations

(a) Show that the adjoint representation of $\mathfrak{su}(2)$ is the $J = 1$. Identify the states $|J = 1, M = \pm 1, 0\rangle$ in terms of the generators.

(b) Consider the representation tensor product $(J = 1) \otimes (J = 1/2)$. Show first that $J_3 |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (m_1 + m_2) |j_1, m_1\rangle \otimes |j_2, m_2\rangle$. Decompose the product space into irreducible subspaces and identify the states.

(c) We normalize the Hilbert space states as $\langle j, \alpha | j, \beta\rangle = \delta_{\alpha\beta}$, where α and β stand for other quantum numbers and j is the highest weight. Show that this implies orthogonality of the other states, i.e. $\langle j - k, \alpha | j - k', \beta\rangle \sim \delta_{\alpha\beta}\delta_{kk'}$.

(d) Within one irreducible representation we use the normalization $\langle j,m|j',m'\rangle = \delta_{mm'}$. Show that the normalization constants in

$$J_-|j-k\rangle = N_{j-k}|j-k-1\rangle\,, \qquad J_+|j-k-1\rangle = N_{j-k}|j-k\rangle$$

are indeed the same.

(e) Convince yourself of the recursion formula

$$N_{j-k}^2 = j-k+N_{j-k+1}^2\,.$$

Show that $N_{j-k} = \frac{1}{\sqrt{2}}\sqrt{(2j-k)(k+1)}$ is a solution with the boundary condition $N_j = \sqrt{j}$.

H 10.3 Complexifications

(a) What is the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$?

(b) Show that $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of $\mathfrak{su}(2)$, i.e. $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}$.

H 10.4 Roots and the Cartan algebra

We consider a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ spanned by the Cartan generators H_i. The remaining generators $E_\alpha \in \mathfrak{g}/\mathfrak{h}$ satisfy $[H_i, E_\alpha] = \alpha_i E_\alpha$. We use the scalar product $\langle A, B\rangle = k\,\mathrm{tr}(A^\dagger B)$. Note that the action considered is always the adjoint, $\mathrm{ad}(A)\cdot B = [A,B]$.

(a) Show that for Hermitean Cartan elements, $H = H^\dagger$, we find that H is self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle$.

(b) Show that $[H_i,[E_\alpha,E_\beta]] = (\alpha_i+\beta_i)[E_\alpha,E_\beta]$.

(c) Show that $[E_\alpha,E_{-\alpha}]$ is in the Cartan algebra. Show further that $[E_\alpha,E_{-\alpha}] = \sum_i \alpha_i H_i$.

(d) Show that for a fixed root α the generators

$$E_\pm = \frac{1}{|\alpha|}E_{\pm\alpha}\,, \qquad E_3 = \frac{1}{|\alpha|^2}\sum_i \alpha_i H_i$$

form a closed and properly normalized $\mathfrak{su}(2)$ subalgebra.