Exercises on Theoretical Particle Physics II

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Due 28/06/2010

9.1 Gauge invariant matter couplings & Super Yang-Mills (*19 credits*)

The way matter couplings are realized in supersymmetric actions will guide us to the **non-abelian** generalization of the gauge invariant action we already studied. Consider a chiral superfields Φ transforming under a global symmetry

$$\Phi \mapsto \Phi' = e^{-i\lambda^a \rho(T_a)}\Phi, \qquad \lambda^a \in \mathbb{R}, \quad a = 1, \dots, \dim(\mathfrak{g}), \tag{1}$$

in a representation[1] ρ of a Lie algebra $\mathfrak{g}$ with generators T_a. In order to gauge this symmetry consistently the transformed superfield Φ' has to remain chiral.

1. Check that (1) respects the chirality of Φ for $\lambda \in \mathbb{R}$ constant and for $\lambda \equiv \Lambda(x, \theta)$ a complete chiral superfield. Althought $W(\Phi)$ can be arranged to be gauge invariant, $\Phi^\dagger\Phi$ cannot. Determine its transformation behaviour. (*2 credits*)

2. In order for this to be gauge invariant introduce a minimal coupling of the vectorsuperfield to the matter contained in the chiral superfield of the form

$$\mathcal{L}_{\text{matter}} \supset \Phi^\dagger e^V \Phi\Big|_{\theta^2\bar\theta^2}, \qquad V = V^a T_a, \quad a = 1, \dots, \dim(\mathfrak{g}). \tag{2}$$

   Determine the right transformation property of e^V for gauge invariance. What is the first order transformation of V? Can you still perform the WZ-gauge? (*3 credits*)

3. Rewrite (2) in the left-chiral represenation by shifting x^μ. This yields $e^{V-2i\theta\sigma^\mu\bar\theta\partial_\mu}$. Why do you expect the covariant derivative? Calculate the D-term ($\theta^2\bar\theta^2$-term) of (2) in the WZ-gauge in the left-chiral representation using $(V_{\text{WZ}})^n = 0$ for $n \geq 3$, thus $e^{V_{\text{WZ}}} = 1 + V_{\text{WZ}} + \frac{1}{2}V_{\text{WZ}}^2$. Identify the covariant derivatives. (*3 credits*)

4. Turning to the kinetic term of the non-abelian gauge sector we have to generalize it further. The non-abelian field strength is defined by

$$W_\alpha = -\tfrac{1}{4}\bar D\bar D\left(e^{-V}D_\alpha e^V\right), \qquad \bar W_{\dot\alpha} = \tfrac{1}{4}DD\left(e^V \bar D_{\dot\alpha} e^{-V}\right). \tag{3}$$

   How does W_α transform under a gauge transformation of e^V? Insert $e^{V_{\text{WZ}}} = 1 + V_{\text{WZ}} + \frac{1}{2}V_{\text{WZ}}^2$ to deduce

$$W_\alpha = -\tfrac{1}{4}\bar D\bar D D_\alpha V + \tfrac{1}{8}\bar D\bar D\,[V, D_\alpha V]\,.$$

   Compare this to the abelian case. Calculate W_α explicitly. You obtain the same result as in Ex. 8.1, (d), replacing ordinary derivatives by covariant ones. (*3 credits*)

[1] In the following we will omit the letter ρ for convenience.

5. Scale the superfield by $V \mapsto 2gV$, where g denotes the **gauge coupling constant**. Next, introduce a complex coupling constant $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$ containing the **theta-angel** Θ and determine the action of the gauge sector given by

$$\mathcal{L}_{\text{gauge}} = \frac{1}{32\pi}\text{Im}\left(\tau \int d^2\theta \text{Tr} W^\alpha W_\alpha\right). \tag{4}$$

Hint: $TrW^\alpha W_\alpha$ is identical to expanded lagrangian for a $U(1)$ guage theory(i.e.check that excersise sheet) with covariant derivatives instead of ordinary ones. Then, multiply this by τ and determine the imaginary part Im. (4 credits)

6. Combine the matter and gauge sector of the action. Integrate out the auxiliary fields to determine the full scalar potential. *(4 credits)*

The result reads

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \\
&= \tfrac{1}{32\pi}\text{Im}(\tau \int d^2\theta \text{Tr} W^\alpha W_\alpha) + \int d^2\theta d^2\bar{\theta}^2 \Phi^\dagger e^{2gV}\Phi + \left(\int d^2\theta W(\Phi) + \text{h.c.}\right) \\
&= \text{Tr}\left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda}\right) + \tfrac{\Theta}{32\pi^2}g^2 \text{Tr} F_{\mu\nu}\tilde{F}^{\mu\nu} \\
&\quad +(D_\mu\varphi)^\dagger D^\mu\varphi - i\psi\sigma^\mu D_\mu\bar{\psi} + i\sqrt{2}g\varphi^\dagger\lambda\psi - i\sqrt{2}g\bar{\psi}\bar{\lambda}\varphi \\
&\quad -\tfrac{1}{2}\frac{\partial^2 W}{\partial\varphi^i\varphi^j}\psi^i\psi^j - \tfrac{1}{2}\frac{\partial^2\bar{W}}{\partial\bar{\varphi}^i\bar{\varphi}^j}\bar{\psi}^i\bar{\psi}^j - V(\varphi^\dagger,\varphi) + \text{total derivatives,}
\end{aligned} \tag{5}$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, the scalar potential

$$V(\varphi^\dagger,\varphi) = F^\dagger F + \tfrac{1}{2}D^2 = \sum_i \left|\frac{\partial W}{\partial\varphi^i}\right|^2 + \frac{g^2}{2}\sum_a \left|\varphi^\dagger T_a\varphi\right|^2 \tag{6}$$

and covariant derivatives

$$\begin{aligned}
D_\mu\lambda &= \partial_\mu\lambda - ig\left[V_\mu^b,\lambda\right], \qquad D_\mu\varphi = \partial_\mu\varphi - igV_\mu^a T_a\varphi, \\
D_\mu\psi &= \partial_\mu\psi - igV_\mu^a T_a\psi, \qquad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig\left[V_\mu,V_\nu\right].
\end{aligned} \tag{7}$$

9.2 Threshold corrections *(17 credits)*

We will consider the renormalization group equations for the three SM couplings at the scale $q^2 < M_{SUSY}^2$, doing a set of manipulations in order to express them in terms of the variations between the SM and the MSSM

$$\delta b_i = b_i^{SM} - b_i, \quad b_i = b_i^{MSSM} \tag{8}$$
$$\Delta_i = \delta b_i \log\frac{q^2}{M_{SUSY}^2}.$$

An estimate of the range at which one can have a SUSY breaking scale T_{SUSY}^2, given a certain scale for the mass of the supertpartners will be performed.
The running of the couplings are given by two sets of equations, one for processes with momenta under the SUSY scale $q^2 < M_{SUSY}^2$

$$\frac{8\pi^2}{g_i^2(q)} = \frac{8\pi^2}{g_i^2(M_{SUSY})} + b_i^{SM}\log\frac{q^2}{M_{SUSY}^2} \tag{9}$$

and another one, for momenta above the SUSY scale $q^2 > M_{SUSY}^2$

$$\frac{8\pi^2}{g_i^2(q)} = \frac{8\pi^2}{g_i^2(M_{GUT})} + b_i \log\frac{q^2}{M_{GUT}^2} \tag{10}$$

(a) Impose continuity for a generic coupling $g^2(q) = g_i^2(q)$ in order to express $8\pi^2/g^2(q)$ under the SUSY scale vs. the parameters Δ, M_{GUT} and b (i.e. to eliminate M_{SUSY}^2 and b^{SM} from Eq.9). Compare the obtained expression with Eq.10. Which role plays Δ? Show it graphically. *(3 credits)*

(b) Specialize the equation obtained in the last part to the three cases $i = 1, 2, 3$. Take into account the relations

$$\begin{aligned}
g_1^2(q) &= \frac{5}{3}e^2(q)/\cos^2(\theta_W), && (11)\\
g_2^2(q) &= e^2(q)/\sin^2(\theta_W), &&\\
g^2(M_{GUT}) &= g_1^2(M_{GUT}) = g_2^2(M_{GUT}) = g_3^2(M_{GUT}), && (12)\\
D &= 5b_1 + 3b_2 - 8b_3. && (13)
\end{aligned}$$

Eliminate $g^2(M_{GUT})$ and θ_W to obtain a single running equation for $\log q^2/M_{GUT}^2$ dependent on the parameters Δ_i, D, and on the functions e^2 and g_3^2. *(2 credits)*

(c) Use the running equation of QCD and the result from previous part to get $8\pi^2/g^2(M_{GUT})$ vs. the parameters b_i, D, Δ_i, and the functions e^2, g_3^2. *(2 credits)*

(d) Obtain now an expression for $\sin^2(\theta_W)$ vs. the parameters Δ_i, b_i, D and the functions e^2, g_3^2. *(2 credits)*

(e) Recall we are working in the range $q^2 < M_{SUSY}^2$, from which we have obtained the running of the couplings in terms of the GUT scale and the corrections $\Delta_i,\ \ i = 1, 2, 3$. Let us now estimate the unknowns $\Delta_1, \Delta_2, \Delta_3$. Is useful to use auxiliary averaged masses M_1, M_2, M_3 such that

$$\begin{aligned}
\sum_\xi b_i^\xi \log(\frac{M_\xi^2}{q^2}) &= -\delta b_i \log(\frac{M_i^2}{q^2}), && (14)\\
\sum_\xi b_i^\xi &= -\delta b_i &&
\end{aligned}$$

and also important

$$\log\frac{M_i^2}{M_{SUSY}^2} \approx 0. \tag{15}$$

Compute the contributions to $sin^2(\theta_W)$ comming from the corrections using the scale M_i^2 for every Δ_i. Express this correction in terms of a single term $A\log(q^2/T_{SUSY}^2)$, giving the constant A and the definition of T_{SUSY}^2 vs. M_1, M_2, M_3. *(4 credits)*

Hint: Use the values $b_1^{SM} = -\frac{41}{10}, b_2^{SM} = \frac{19}{6}, b_3^{SM} = 7$ *and* $b_1 = -\frac{66}{10}, b_2 = -1, b_3 = 3$ *for the coefficients of the* log *in the SM and MSSM respectively.*

(f) The term including the scale of SUSY breaking T_{SUSY}^2 gives the deviation of the renormalization group equation for $sin^2(\theta_W)$ in the SM, with respect to the one in the MSSM. Take as input the parameters b_i^ξ, M_ξ^2 defined in Eq.14, to give finally

$$\begin{aligned} -19 \log \frac{q^2}{T_{SUSY}^2} &= 3 \log \frac{q^2}{M_{squarks}^2} + 28 \log \frac{q^2}{M_{gluino}^2} \\ &- 3 \log \frac{q^2}{M_{slepton}^2} - 32 \log \frac{q^2}{M_{wino/zino}^2} - 12 \log \frac{q^2}{M_{Higgsino}^2} \\ &- 3 \log \frac{q^2}{M_{Higgs}^2} \end{aligned} \tag{16}$$

Consider the mass of the supertpartners situated at a given scale $10^k GeV$. Do the redefinition $A_\alpha = M_{squarks}^2, M_{gluino}^2 ..., \alpha = 1, ...6$, with $A_\alpha = 10^k GeV \tilde{a_\alpha}$ and $1 \leq \tilde{a_\alpha} < 10$. Find an expression of T_{SUSY}^2 vs. the A_α, and obtain the maximum and the minimum bound for the SUSY breaking scale. (*4 credits*)