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Theoretische Physik

Exercises on String Theory II

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–IN-CLASS EXERCISES–
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On this exercise sheet we want to explore some features of orbifold compactifications. Given that holonomy is a crucial ingredient when considering spaces (manifolds) which could be physically meaningful, we devote the first exercise to studying parallel transport in two very simple examples.

As shown in the lecture, orbifolds are successful grounds to obtain $\mathcal{N} = 1$ SUSY in 4D. Since they are flat everywhere but at the singularities, the spectrum and many other physical quantities can be exactly computed. In the second exercise we consider the $\mathbb{T}^6/\mathbb{Z}_3$ orbifold. Using the standard embedding, we attempt to find out which is the corresponding gauge group for such model.

Exercise 0.1: Examples of Holonomy

Consider first the sphere $S^2 : (x^0, x^1, x^2) : (x^0)^2 + (x^1)^2 + (x^2)^2 = 1$. A coordinate chart on $S^2\backslash\{(1, 0, 0)\}$ is given by the map $x^a(r, \phi)$:

$$x^0 = \frac{r^2 - 1}{r^2 + 1}, \quad x^1 = \frac{2r\cos\phi}{r^2 + 1}, \quad x^2 = \frac{2r\sin\phi}{r^2 + 1} \tag{1}$$

which corresponds to a stereographic projection from the north pole onto a plane through an equator. The embedded metric satisfies

$$ds^2 = \frac{4}{(1 + r^2)^2}(dr^2 + r^2 d\phi^2)$$

(a) Compute the Christoffel symbols.
Hint: you could make use of the Euler Lagrange formalism.

(b) Write the parallel transport equation for an arbitrary vector T^a $(a = r, \phi)$ along a curve $(r(t), \phi(t))$. How does the vector change when parallel transported along the curve $r = R$ (constant), from a point ϕ_0 to $\phi_0 + \alpha$. For which value of R does this curve correspond to a geodesic? Note that in general $T^a(\phi_0) \neq T^a(\phi_0 + 2\pi)$. Argue why this result leads to the conclusion that the sphere has non trivial holonomy? Rewrite $T = (T^r, RT^\phi)$ as $T(\phi_0 + 2\pi) = MT(\phi_0)$. Compute M. Which group does this sort of transformations belong to?

Consider now the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. The lattice spanned by the roots of SU(3)

$$\alpha^1 = \sqrt{2}, \quad \alpha^2 = \frac{-1+\sqrt{3}\,\mathrm{i}}{\sqrt{2}}, \tag{2}$$

has a $\mathbb{Z}_3$ isometry and hence it suits to construct the orbifold of our interest. The generator θ of the point group is just a rotation by $2\pi/3$ around the origin.

(c) What is the action of θ on the basis vectors? What are the fixed points within the fundamental domain of the torus? What is the fundamental domain of the orbifold? What does the orbifold look like?

(d) Recall that the orbifold is flat everywhere but at the fixed points. Thus, it is trivial to transport a vector along curve which does not pass through any of the singularities. Which kind of closed loops do not leave a vector invariant after parallel transport? How do the vectors look after being transported along such loops? What is the "holonomy" of the orbifold?

Exercise 0.2: Gauge group of the $\mathbb{Z}_3$ orbifold in Standard Embedding

In this exercise we want to explore how the gauge group of the heterotic theory gets broken upon orbifolding. The boundary conditions for the extra 16 left moving coordinates are of the form

$$X^I(\tau+\sigma+\pi) = X^I(\tau+\sigma) + \pi p^I \tag{3}$$

with $p \in \Gamma_{16}$.

(a) Find a mode expansion consistent with (3).

(b) Modular invariance of the one loop partition function requires Γ_{16} to be either the lattice of $\mathrm{E}_8 \times \mathrm{E}_8$ or that of $Spin(32)/\mathbb{Z}_2$. In the fist case, one can make use of the direct product structure and consider a single E_8. Vectors in such lattice have the form

$$(n_1, n_2, ..., n_8), \quad \sum_{i=1}^{8} n_i = 0 \mod 2$$
$$\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}, ..., n_8+\frac{1}{2}\right), \quad \sum_{i=1}^{8} n_i = 0 \mod 2$$

where $n_1, n_2, ..., n_8 \in \mathbb{Z}$. Consider only contributions coming from the gauge coordinates. Use the mass equation

$$\frac{m_L^2}{4} = \frac{p^2}{2} + N - 1 \tag{4}$$

(N is the oscillator number), to find the massless states which originate from the gauge sector.

(c) In order to preserve the modular properties of the theory, in general the orbifold rotation needs to act also on the gauge coordinates. One choice for the action is the standard emebedding:

$$X^I \xrightarrow{\theta} X^I + t^I$$

with

$$t^I = \frac{1}{3}(1, 1, -2, 0, ..., 0).$$

The effect of such an embedding is that the only states from the $\mathcal{N} = 1$ vector multiplet which satisfy

$$p \cdot t = 0 \mod 1\,.$$

are present in the physical spectrum. Restrict to the first E_8 and find the massless weights which satisfy the previous projections. Split them into two sets which are mutually orthogonal. How many of them are in each set? Does this tell you something about the groups each of these sets is associated to?

You can convince yourself that the gauge group after orbifolding is $E_6 \times SU(3) \times E_8$ by looking at the simple roots in each set, and computing the Dynkin diagrams. Note that the resulting group has the same rank as $E_8 \times E_8$. Why is that so?