Exercises on String Theory II

Prof. Dr. H.P. Nilles, Priv. Doz. Dr. S. Förste

–Home Exercises–
To be discussed on 05 June 2012

In the exercise we further analyze the conformal group. The first exercise deals with the conformal group in 2 dimensions. We will find that the global conformal transformations in 2 dimensions are precisely the Möbius transformations. In the second exercise we will calculate the two-point correlator for free bosons.

Exercise 5.1: The conformal group in $d = 2$ dimensions (*10 credits*)

On the last exercise sheet, you showed that for infinitesimal conformal transformations $x^\mu \mapsto x^\mu + \epsilon^\mu$ one obtains

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\,. \tag{1}$$

(a) Show that in two dimensions, any holomorphic function gives rise to an infinitesimal conformal transformation in some open set. (*2 credits*)

Extending the trafos to meromorphic functions, the conformal transformations can be written in a Laurent expansion as

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} (-z^{n+1})\epsilon_n\,, \tag{2}$$

$$\overline{z}' = \overline{z} + \overline{\epsilon}(\overline{z}) = \overline{z} + \sum_{n\in\mathbb{Z}} (-\overline{z}^{n+1})\overline{\epsilon}_n\,, \tag{3}$$

with constant infinitesimal parameters $\epsilon_n\,, \overline{\epsilon}_n$. The corresponding generators are

$$l_n = -z^{n+1}\partial_z\,, \qquad \bar{l}_n = -\overline{z}^{n+1}\partial_{\overline{z}}\,. \tag{4}$$

(b) Calculate the commutators $[l_m, l_n]\,, [\bar{l}_m, \bar{l}_n]\,,$ and $[l_m, \bar{l}_n]\,.$ (*3 credits*)

(c) The l_m are not well–defined at $z = 0$ on $\mathbb{C}$. Going to $S^2 = \mathbb{C} \cup \{\infty\}$ improves on this point, but there are still generators which are not globally defined at $z = 0$ and $z = \infty$. Argue that only $\{l_{-1}, l_0, l_1\}$ are globally defined. (*2 credits*)

(d) Argue that (*3 credits*)

   (i) l_{-1} generates translations,

   (ii) $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ generate dilations and rotations, respectively,

   (iii) l_1 generates special conformal transformations $z \mapsto \frac{z}{cz+1}$.

Together, these transformations form the group $PSL(2,\mathbb{C})$, which you encountered on exercise sheet 10 last year.

Exercise 5.2: Two point function for free bosons ***(10 credits)***

The action for a free boson reads

$$S = \frac{1}{4\pi g}\int dz d\bar{z}\, \partial X^\mu \overline{\partial} X_\mu \tag{5}$$

with $\partial \equiv \partial_z \,, \overline{\partial} \equiv \partial_{\overline{z}}$.

(a) Show that the action is invariant if X has conformal dimension $(0,0)$. (*2 credits*)

(b) Calculate the equations of motion for X. What do they imply? (*1 credit*)

(c) Next we want to calculate the correlator $\langle X^\mu(z,\overline{z}), X^\nu(z',\overline{z}')\rangle$. Look at the path integral variation

$$\int \mathcal{D}X \frac{\delta}{\delta X_\mu(z,\overline{z})}\left[e^{-S}\, X^\nu(z',\overline{z}')\right] \tag{6}$$

to derive (*2 credits*)

$$\langle \eta^{\mu\nu}\delta(z-z',\overline{z}-\overline{z}')\rangle = -\frac{1}{2\pi g}\partial\overline{\partial}\,\langle X^\mu(z,\overline{z})X^\nu(z',\overline{z}')\rangle\,. \tag{7}$$

We claim that the Green's function G solving $\partial\overline{\partial}\, G = -2\pi g\, \delta(z-z',\overline{z}-\overline{z}')$ is given by $G = -g\ln(|z-z'|^2)$.

(d) To prove this, first argue that the equation holds for $z \neq z'$. (*1 credit*)

(e) To argue that G indeed gives rise to a delta function, we define $w := z - z'$ and consider $\int_U \partial\overline{\partial}\ln|w|^2\, dw\, d\overline{w}$ with U some open set containing 0. Perform the integral to confirm the claim. Note that this confirms our previous result that X has conformal dimension $(0,0)$. (*4 credits*)
Hint: Use Stoke's theorem to rewrite the integral as a contour around 0. Then invoke the residue theorem to perform the integration or just evaluate it by hand.