Exercises on String Theory II

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To be discussed on 21 June 2012

After we have obtained the two-point correlation function of the free boson on Exercise sheet 5, we now calculate the two-point correlation function and the conformal weights of a free fermion. In the second exercise, we examine the linear dilaton CFT, which is an extension of the CFT of a free boson with a coupling to the worldsheet gravity.

Exercise 7.1: Two-point function for free fermions ***(14 credits)***

The action for a free Majorana fermion reads

$$S = \frac{1}{4\pi g} \int dx^0 dx^1 \sqrt{|h|}\,(-i)\overline{\Psi}\gamma^\alpha \partial_\alpha \Psi \,, \tag{1}$$

where g is a constant, $\overline{\Psi} = \Psi^\dagger \gamma^0$, $h_{\alpha\beta} = \mathrm{diag}(1,-1)$, and the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

(a) What is the Majorana condition on the components ψ, $\overline{\psi}$ of Ψ? (*1 credit*)

(b) Perform a Wick rotation $x_1 \mapsto ix_1$ and define $z := x^0 + ix^1$ to rewrite the action as

$$S = \frac{1}{4\pi g} \int dz d\overline{z} \left(\psi(z,\overline{z})\overline{\partial}\psi(z,\overline{z}) + \overline{\psi}(z,\overline{z})\partial\overline{\psi}(z,\overline{z}) \right) . \tag{3}$$

(*3 credits*)

(c) Calculate the equations of motion for ψ and $\overline{\psi}$. What do they imply? (*1 credit*)

(d) By imposing invariance of the action (3) under conformal transformations, calculate the conformal weights $(h,\overline{h})$ of ψ and $\overline{\psi}$. (*2 credits*)

(e) Next we want to calculate the correlator $\langle \Psi_i(z,\overline{z}), \Psi_j(z',\overline{z}') \rangle$ where $i,j = 1,2$ label the components of Ψ. To do so, express the kinetic terms of the components in (3) as a matrix A_{ij} and write down the differential equation for the Green's function. (*2 credits*)

We claim that the Green's function $G_{ij}(z, z')$ for the equation obtained in (e) is given by

$$G = 2g \begin{pmatrix} \overline{\partial}\, \frac{1}{z-z'} & 0 \\ 0 & \partial\, \frac{1}{\overline{z}-\overline{z}'} \end{pmatrix} , \tag{4}$$

(f) Prove this using the techniques you already used in Exercise 5.2 (e) for the bosonic case. *(5 credits)*

Exercise 7.2: The Linear Dilaton CFT ***(6 credits)***

In this theory, a linear dilaton term $\Phi(X)$ is introduced which couples the boson X to worldsheet gravity. The corresponding action is

$$S_\Phi = \int d^2\sigma\, \Phi(X) R^{(2)} \tag{5}$$

with $\Phi(X) = QX$ where Q is a constant. For a flat worldsheet, the quantization of X proceeds as before. The linear dilaton coupling shows up in the energy-momentum tensor, which reads

$$T_Q(z) = \frac{1}{2}\Big(:\partial X(z) \partial X(z): + \widetilde{Q}\, \partial^2 X(z) \Big) \tag{6}$$

where $\widetilde{Q}$ is a constant related to Q.

Calculate the OPE of T_Q with itself, $T_Q(z)\, T_Q(w)$. Show that it has the correct form and read off the central charge. *(6 credits)*
Hint: For $:\partial X(z) \partial X(z): \; :\partial X(w) \partial X(w):$ *you may use the result from the lecture. Use Wick's theorem and the result from Exercise sheet 5 to evaluate the other contractions.*