Exercises on String Theory II

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–Home Exercises–
To be discussed on 28 June 2012

Exercise 8.1: Some final remarks on the free boson CFT ***(9 credits)***

The expansion of the free massless scalar X living on a Lorentzian cylinder is given by

$$X(\tau,\sigma) = x + 4p\tau + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n \mathrm{e}^{2\mathrm{i}n\sigma} + \tilde{\alpha}_n \mathrm{e}^{-2\mathrm{i}n\sigma}\right)\mathrm{e}^{-2\mathrm{i}n\tau}\,, \tag{1}$$

where $\sigma \simeq \sigma + \pi$ and for convenience we have taken $\alpha' = 2$.

(a) Perform the standard Wick rotation $\tau \to -\mathrm{i}\tau$ and then, similarly as in exercise 6, express the expansion (1) in terms of the complex coordinates *(2 credits)*

$$z = \mathrm{e}^{2(\tau - \mathrm{i}\sigma)}, \quad \bar{z} = \mathrm{e}^{2(\tau + \mathrm{i}\sigma)}\,.$$

(b) The only non-trivial commutation relations between x, p, α_n, $\tilde{\alpha}_n$ are given by

$$[x,p] = \mathrm{i}, \quad [\alpha_n, \alpha_m] = [\tilde{\alpha}_n, \tilde{\alpha}_m] = m\delta_{m+n,0}\,.$$

Define normal ordering via

$$:xp: \;=\; :px: \;=\; xp\,, \tag{2}$$
$$:\alpha_m\alpha_{-n}: \;=\; :\alpha_{-n}\alpha_m: \;=\; \alpha_{-n}\alpha_m\,, \tag{3}$$
$$:\tilde{\alpha}_m\tilde{\alpha}_{-n}: \;=\; :\tilde{\alpha}_{-n}\tilde{\alpha}_m: \;=\; \tilde{\alpha}_{-n}\tilde{\alpha}_m\,, \tag{4}$$

with $n, m > 0$. Use the expansion you found in (a) and the previous prescription to show that the following relation holds *(4 credits)*

$$X(z,\bar{z})X(w,\bar{w}) \;=\; :X(z,\bar{z})X(w,\bar{w}): -\log|z-w|^2 \tag{5}$$

Hint: You may need to use the indentity

$$\sum_{n>0}\frac{x^n}{n} = -\log(1-x)$$

(c) Finally, use what you learned in the lecture to prove the following OPE *(3 credits)*

$$\partial X(z)\;:\mathrm{e}^{\mathrm{i}\alpha X(w)}: \;= -\frac{\mathrm{i}\alpha}{z-w}:\mathrm{e}^{\mathrm{i}\alpha X(w)}: +\mathrm{reg.} \tag{6}$$

Exercise 8.2: The bc-System ***(11 credits)***

Consider the Polyakov action of the bosonic string. While the action is explicitly invariant under reparametrizations and Weyl transformations, the path integral measure is not. To alleviate this problem one can apply the Faddeev-Popov method, which effectively results in a ghost sector being introduced. The action for such a sector is described by

$$S_G = \frac{1}{2\pi} \int d^2z \left(b\bar{\partial}c + \bar{b}\partial\bar{c} \right) \tag{7}$$

(a) Find the equations of motion for the ghost and anti-ghost fields. (*1 credit*)

(b) Use a method similar to the one of exercise 7.1 (e) to prove the following OPEs (*3 credits*)

$$\langle c(z)b(w)\rangle = \frac{1}{z-w}\,, \tag{8}$$

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}\,. \tag{9}$$

Use these results to argue that b and c are fermionic fields.

(c) The energy-momentum tensor for the ghost system is given by

$$T_G = -2 : \partial c(z) b(z) : + : c(z)\partial b(z) : \,. \tag{10}$$

Compute the OPEs which are relevant to show that the fields $c(z)$ and $b(z)$ are primary fields with conformal weights $h=-1$ and $h=2$, respectively. The fields $c(z)$ and $b(z)$ have different conformal weights but their two-point funtion is non-zero. Due to which property of these fields does the result from exercise 6.2 (d) not apply here? (*4 credits*)

(d) Compute the OPE for $T(z)T(w)$ and use it to deduce the central charge for the ghost system. (*3 credits*)

Side remark:
We know that at the classical level the trace of the energy-momentum tensor in a CFT must vanish. Upon quantization it is found that the expectation value for this trace is given by

$$\langle T^\alpha_\alpha \rangle = -\frac{c}{12} R\,, \tag{11}$$

where R is the Ricci scalar. This result means that Weyl invariance is only guaranteed at the quantum level if $\langle T^\alpha_\alpha\rangle$ is equal to zero. The Weyl anomaly can be cancelled only if $R=0$ (which is not of much interest, since then we are only allowed to work in flat space) or if $c=0$.