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Theoretische Physik

Exercise 1
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SS 13

Exercises on Theoretical Particle Astrophysics

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Among many important applications, Lie algebras and Lie groups are used to describe gauge interactions in particle physics models. This exercise sheet is devoted to study the Lie algebra of a particular class of those, namely special unitary ($SU(N)$) groups. As you already know, $SU(2)$ plays an important role in the description of spin $\frac{1}{2}$ particles as well as the weak interactions, $SU(3)$ is compulsory for quantum chromodynamics and $SU(5)$ is a very popular alternative for a grand unified theory (GUT), just to cite some examples. In the first exercise we attempt to make contact with the intuitive picture and for that we take the simplest example $SU(2)$. The second exercise deals with the general case. Selected references on group theory can be found at the end of the sheet.

1.1 The Lie algebra of $SU(2)$ — *9 points*

Consider the group

$$\mathrm{SU}(2) := \left\{ U \in \mathrm{GL}(2,\mathbb{C}) \;\middle|\; U^\dagger = U^{-1}\ ,\ \det U = 1 \right\} .$$

(a) Show that SU(2) as a manifold is equivalent to a 3-sphere ($\mathrm{SU}(2) \cong S^3$). (*3 points*)
Hint: Find an equation constraining the parameter space of U to S^3 as a submanifold in $\mathbb{R}^4$.

The previous exercise shows that SU(2) is an example of a **Lie group**, i.e. a group which admits the structure of a differentiable manifold.

(b) Introduce spherical coordinates on S^3 to infer

$$U(\omega,\theta,\phi) = \cos(\omega)\cdot \mathbb{1} \;+\; \mathrm{i}\sin(\omega)\cdot(\vec{\omega}_0\cdot\vec{\sigma}) \tag{1}$$

where $\vec{\omega}_0 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T \in S^2$ and the σ_i are the Pauli matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(*3 points*)

(c) Consider the three dimensional vector $\vec{\omega} \equiv \omega \cdot \vec{\omega}_0$. What is its parameter space? What can you say about its boundary? After the proper identification one can see that the space defined by $\vec{\omega}$ is topologically equivalent to S^3. *(1 point)*

(d) Use eq. (1) to show that every $U \in SU(2)$ obeys the differential equation

$$\frac{\partial U}{\partial \omega} = \mathrm{i}(\vec{\omega}_0 \cdot \vec{\sigma})U \, .$$

Integrate this to determine the solution

$$U(\vec{\omega}) = \exp\{\mathrm{i}\vec{\omega} \cdot \vec{\sigma}\} \, ,$$

compute the following quantities:

$$T_i = \left.\frac{\partial U}{\partial \omega^i}\right|_{\vec{\omega}=0} \quad \text{for } i = 1, 2, 3$$

and finally show that they satisfy the commutation relation $[T_i, T_j] = 2\mathrm{i}\epsilon_{ijk}T_k$. *(2 points)*

Even though it was a very simple example, the previous exercise shows that by introducing a set of coordinates and by using the differentiability of the manifold, we can parameterize any element of $SU(2)$ in terms of very local data (the tangent vectors T_i).
In general, given the coordinates x^i for a Lie group G, we can define the basis

$$T_i := \left.\frac{\partial g}{\partial x^i}\right|_{g=\mathbb{1}}$$

for the tangent space $\mathfrak{g} := T_{\mathbb{1}}G$ at the identity element $\mathbb{1} \in G$. The vectors T_i are called the **generators of the Lie algebra** $\mathfrak{g}$. It is known that in a certain vicinity of the identity, the elements of G can be written in the form $e^{\mathrm{i}x^i T^i}$, as we can see in the case of $SU(2)$.

In more formal terms, a Lie algebra is vector space with a binary operator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Given $a, b, c \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$ the operator must satisfy

- $[\lambda a, b] = \lambda[a, b]$ (linear),
- $[a, b] = -[b, a]$ (skew-symmetric),
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ (Jacobi identity).

Note that in the previous $SU(2)$ example we have taken the bilinear $[\cdot, \cdot]$ to be the standard commutator.

2.1 Roots, Cartan matrix and Dynkin diagram of $\mathfrak{su}(N)$ *12 points*

Consider the space of all $N \times N$ matrices and regard it as a Lie algebra $\mathfrak{gl}(N)$. We choose as a basis the elements e_{ab} with components $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

(a) Verify the multiplication rule and thus the commutator operation on the algebra (*1 point*)

$$e_{ab}e_{cd} = e_{ad}\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad}.$$

(b) Let us now restrict to the case of $SU(N)$. Take an arbitrary element $U = e^{\mathrm{i}M}$. Which properties does M need to satisfy? Use this result to write a basis for the generators of $\mathfrak{su}(N)$. What is the dimension of the algebra? (*2 points*)
(*Hint:* $\det e^M = e^{\operatorname{tr} M}$)

(c) The **Cartan algebra** $\mathfrak{h}$ is defined as the maximal commuting subalgebra of the Lie algebra. Its dimension is called the **rank** of the Lie algebra. Give a possible choice for the Cartan subalgebra of $\mathfrak{su}(N)$. What is the rank r of $\mathfrak{su}(N)$? (*1 point*)

(d) Now we want to diagonalize the Cartan algebra in the adjoint representation, which acts by the commutator

$$\operatorname{ad} h(g) = [h, g]$$

Perform a (complex) basis change of $\mathfrak{su}(N)/\mathfrak{h}$ to an eigenbasis of $\mathfrak{h}$. You should find,

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab}\,, \tag{2}$$

with $h = \sum_i \lambda_i e_{ii}$. (*1 point*)

We can regard eq. (2) (for e_{ab} fixed) as a prescription for how to associate a number $(\lambda_a - \lambda_b)$ to each $h \in \mathfrak{h}$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b.$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra $\mathfrak{h}$. This dual space is commonly denoted by $\mathfrak{h}^*$.
Let $\alpha_1 \ldots \alpha_r$ be a fixed basis of roots so every element of $\mathfrak{h}^*$ can be written as $\rho = \sum_i c_i\alpha_i$. We call ρ **positive** ($\rho > 0$) if the first non-zero coefficient c_i is positive. Note, that the basis roots α_i are positive by definition. If the first non-zero coefficient c_i is negative, we call ρ negative. For $\rho, \sigma \in \mathfrak{h}^*$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$. A **simple root** is a positive root which can not be written as the sum of two positive roots.

(d) Now choose a basis α_i for the root space of the form

$$\alpha_i(h) = \lambda_i - \lambda_{i+1}, \quad i = 1, 2, \ldots, N-1\,.$$

Verify that these roots are a basis and that they are positive with $\alpha_1 > \alpha_2 > \ldots > \alpha_{N-1}$. Show that these roots are simple roots. (*1.5 points*)

Next, we define a structure that resembles a scalar product on the algebra. Let t_i be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra:

$$[x, [y, t_i]] = \sum_j K_{ij} t_j\,.$$

The **Killing form** is then defined as $\mathcal{K}(x, y) := \operatorname{Tr}(K)$.

(e) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h, h')$, where $h = \sum_i \lambda_i e_{ii}$, $h' = \sum_j \lambda'_j e_{jj}$. *(1.5 points)*

The Killing form enables us to make a connection between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$: One can prove that if $\alpha \in \mathfrak{h}^*$, there exists a unique element $h_\alpha \in \mathfrak{h}$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \quad \forall h \in \mathfrak{h}\,.$$

(f) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find h_{α_i} from comparison with your result from (e). *(1 point)*

With the help of the h_α, we are now able to define a scalar product on $\mathfrak{h}^*$:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \quad \text{where } \alpha_i, \alpha_j \in \mathfrak{h}^*.$$

(g) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}\,.$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root α_i, we associate a small circle and join the small circles i and j with $A_{ij}A_{ji}$ (no summation, $i \neq j$) lines. *(1.5 points)*

(h) Draw the Dynkin diagram for $\mathfrak{su}(N)$. *(0.5 points)*

References

[1] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations.* Cambridge University Press, 2003.

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[3] C. Luedeling, *Group Theory for Physicists*, Lecture Notes, Bonn, SS 2010. `http://www.th.physik.uni-bonn.de/nilles/people/luedeling/grouptheory/data/grouptheorynotes.pdf` Marina von Steinkirch

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