Physikalisches Institut
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Theoretische Physik

Exercise 6
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SS 13

Exercises on Theoretical Particle Astrophysics

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–Home Exercises–
Due 31st May

In this exercise we want to discuss the cosmological evolution. In order to describe the dynamics of space-time we consider some basic concepts of general relativity (GR).

6.1 Friedmann-Robertson-Walker Cosmology *20 points*

A four dimensional *homogeneous* and *isotropic* universe is described by the **Robertson–Walker metric** taken from the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right) .$$

Where k is a curvature parameter that specifies whether the universe is open ($k < 0$), flat ($k = 0$) or closed ($k > 0$).

(a) Read of the metric $g_{\mu\nu}$ and calculate its inverse $g^{\mu\nu}$. (*2 points*)

(b) Calculate the **Christoffel symbols** $\Gamma^\lambda_{\mu\kappa}$ from the metric defined as

$$\Gamma^\lambda_{\mu\kappa} = \frac{1}{2}g^{\lambda\nu}\left(\frac{\partial g_{\nu\mu}}{\partial x^\kappa} + \frac{\partial g_{\nu\kappa}}{\partial x^\mu} - \frac{\partial g_{\mu\kappa}}{\partial x^\nu}\right) . \tag{1}$$

You should find

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2} \quad \Gamma^0_{22} = a\dot{a}r^2 \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \\
&\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \\
&\Gamma^1_{22} = -r(1-kr^2) \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \\
&\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \\
&\Gamma^2_{33} = -\sin\theta\cos\theta \quad \Gamma^3_{23} = \cot\theta
\end{aligned}$$

Note that there is a symmetry in the lower two components! (*4 points*)

(c) Now consider the **Ricci-Tensor** which is derived from the Christoffel symbols as

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\lambda} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} \,. \tag{2}$$

Convince yourself that the Ricci tensor is (always) symmetric and calculate the components for the RW metric. You should find *(3 points)*

$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right] g_{ij} \,.$$

The energy momentum tensor of the fields in the universe respecting its symmetries is the one of a perfect fluid

$$T^\mu_{\ \nu} = \mathrm{diag}(-\rho(t), p(t), p(t), p(t))$$

with the energy density $\rho(t)$ and the pressure density $p(t)$.

(d) Compute the curvature scalar $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$. Write down the 00 and the ii components of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = 8\pi G T_{\mu\nu} \,.$$

The 00 component is also called **Friedmann equation**. *(2 points)*

(e) Derive the first law of thermodynamics *(1 point)*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) = -p\frac{\mathrm{d}}{\mathrm{d}t}a^3 \,.$$

Now we have two independent equations but yet three independent functions, therefore we need another equation to get the chance of finding a solution. We employ the equation of state $p(t) = w\rho(t)$, where w depends on the specific properties of the fluid. One finds

- $w = 0$ for static matter
- $w = 1/3$ for radiation (due to $T^\mu_{\ \mu} = 0$)
- $w = -1$ for vacuum energy (due to $T_{\mu\nu} \propto g_{\mu\nu}$) .

(f) How does the energy density change with the radius in these three cases? *(1 point)*

(g) Define the **Hubble parameter** $H(t) := \frac{\dot{a}(t)}{a(t)}$,the critical density $\rho_C := \frac{3H^2}{8\pi G}$ and $\Omega := \frac{\rho}{\rho_C}$ and rewrite the Friedmann equation as

$$\frac{k}{H^2a^2} = \Omega - 1 \,.$$

In an expanding universe, how does the type of the universe (closed/flat/open) depend on the energy density? *(2 points)*

(h) We introduce the notation with an index zero for the todays values for the quantities a, Ω, H ... . Rewrite the Friedmann equation again as

$$\left(\frac{\dot{a}}{a_0 H_0}\right)^2 = 1 - \Omega_0 + \Omega_0 \left(\frac{a_0}{a}\right)^\alpha$$

with $\alpha = 3w + 1$. *(1 point)*

(i) We define the time t such that $a(t = 0) = 0$. This allows us to compute the age of the universe

$$t \equiv \int_0^{a_0} \frac{\mathrm{d}a}{\dot{a}} \,.$$

Show that

$$t = H_0^{-1} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - \Omega_0 + \Omega_0 x^{-\alpha}}}$$

and compute the age of matter- and radiation dominated universes in the open, flat and closed case. *(4 points)*