Exercises on Theoretical Particle Physics II

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5. Grassmann variable integration (*5 credits*)

The Grassman integration is defined as

$$\int d\theta^\alpha := 0 \text{ and } \int d\theta^\alpha \theta_\beta := \delta^\alpha_\beta, \tag{1}$$

which is linear. Note in particular that integration and differentiation is the same operation for Grassmann variables. The volume elements are

$$d^2\theta := -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\bar\theta := -\frac{1}{4} d\bar\theta_{\dot\alpha} \bar\theta_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}, \quad d^4\theta := d^2\theta d^2\bar\theta. \tag{2}$$

From this, we find

$$\int d^2\theta(\theta\theta) = \int d^2\bar\theta(\bar\theta\bar\theta) = 1. \tag{3}$$

(a) Owing to the nilpotence of Grassmann variables the Taylor series expansion of any function $f(\theta, \bar\theta)$ is finite. Write down the Taylor expansions of $f(\theta)$ and $f(\theta, \bar\theta)$ and determine $\int d^2\theta f(\theta)$ and $\int d^4\theta f(\theta, \bar\theta)$ in terms of their Taylor series expansion coefficients. Expressions like these appear in the SUSY action when formulated in superspace. (*1 credit*)

(b) Let us define

$$I := \int \mathrm{d}^N x \exp[-x_j A_{jk} x_k], \tag{4}$$

with an $N \times N$ matrix A. Assume A is symmetric. Prove $I = \left(\frac{\pi^N}{\det(A)}\right)^{\frac{1}{2}}$ where the integral runs from $-\infty$ to ∞. (*2 credits*)

(c) Now we replace the 'bosonic' coordinate x_i by complex Grassmann variables θ_i and let A be a generic matrix. Compute

$$\int \mathrm{d}^N\theta \mathrm{d}^N\bar\theta \ \exp[-\bar\theta_j A_{jk} \theta_k]\,.$$

(*2 credits*)

6. The Superpotential (*5 credits*)

In this and the next excercise we learn how to construct SUSY invariant actions. Starting with a left-chiral superfield ϕ, we can write down the most general SUSY invariant action as

$$S = \int d^4x \left[\int d^2\theta d^2\bar\theta K(\phi, \phi^\dagger) + \int d^2\theta W(\phi) + \text{h.c.}\right]. \tag{5}$$

The function K is the **Kähler potential** and W is the **superpotential**. The hermitian conjugate is added in order to make the expression real. In this exercise we check that the superpotential is indeed SUSY invariant and investigate its structure. We know that the expansion of a field in θ is finite. The left-chiral superfield ϕ is commonly expanded as:

$$\phi_L(x, \theta) = \varphi(x) + \sqrt{2}\theta\psi(x) + F(x)\theta\theta, \tag{6}$$

where φ and F are bosonic fields and ψ is fermionic.

(a) Show that the part containing the superpotential in (5) is indeed SUSY invariant up to a total derivative (why doesn't this matter in the action?). To show this, use the infinitesimal SUSY trafo $\delta_{(\epsilon,\bar{\epsilon})} = \epsilon Q + \bar{Q}\bar{\epsilon}$, with Q and $\bar{Q}$ in left-chiral representation. (*2 credits*)

(b) Let us take the superpotential

$$W = m\phi^2 + \lambda\phi^3. \tag{7}$$

Calculate all occuring terms in the left-chiral representation using (6).(*3 credits*)

7. The Kähler potential (*10 credits*)

In this exercise we look at the simplest Kähler potential $K = \phi\phi^\dagger$ and investigate its properties.

(a) Show that the part containing K in (5) is also SUSY invariant up to a total derivative. (*2 credits*)

(b) Show that $[\phi_L(x, \theta)]^\dagger$ transforms in the right-chiral representation of the SUSY algebra. (*2 credits*)

(c) Use the relations between S, S_L, and S_R from excercise 4 to determine the relations between ϕ, ϕ_L, and ϕ_R. (*3 credits*)

(d) Calculate all terms in S coming from $K = \phi\phi^\dagger$. Use again the left-chiral representation. (*3 credits*)

Hint: If you only consider terms which survive the integration the excercise simplifies.