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The Bosonic String

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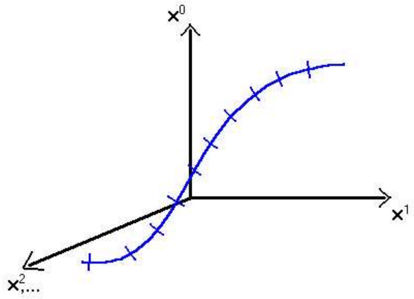
1 Motivation

The Relativistic Particle

- Free relativistic particle (mass m) moving in d-dimensional Minkowski space with $\eta_{\mu\nu} = \text{diag}(- + ...+)$



- $x^\mu(\tau)$, $\mu = 1, ..., d$ describe the embedding of the point particle in space-time

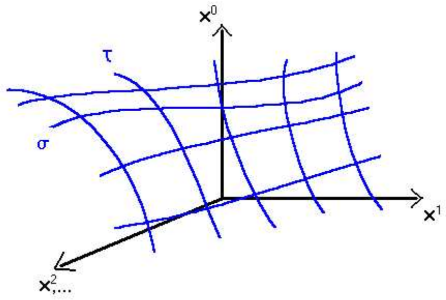
- Action $\hat{=}$ length of the world-line

$$
\begin{aligned}
S &= -m \int_{s_0}^{s_1} \mathrm{d}s \\
&= -m \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \left[-\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \eta_{\mu\nu} \right]^{1/2}
\end{aligned}
$$

2 The Relativistic String

The Nambu-Goto Action

- Free string, described by parameters $\sigma^\alpha = (\sigma, \tau)$
  τ: proper time, σ: $0 \leq \sigma < \bar{\sigma}$
  with: $\bar{\sigma} = \pi$ for open string, $\bar{\sigma} = 2\pi$ for closed string



- $X^\mu(\sigma, \tau)$, $\mu = 1, ..., d$ real functions describe the embedding of the world-sheet in the d-dimensional space
  Notation: $\dot{X} \equiv \frac{\partial X}{\partial \tau}$ and $X' \equiv \frac{\partial X}{\partial \sigma}$

- In analogy: action $\hat{=}$ area of the world-sheet

$$
\begin{aligned}
S_{NG} &= -T \int \mathrm{d}A \\
&= -T \int \mathrm{d}^2\sigma \, \left[-\det \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \right]^{1/2} \\
&= -T \int \mathrm{d}^2\sigma \, \left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{1/2} \\
&= -T \int \mathrm{d}^2\sigma \, \sqrt{-\det \Gamma_{\alpha\beta}}
\end{aligned}
$$

  Notation $\Gamma_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

- String tension T = constant, $[T] = \text{mass}^2$

- Disadvantage: square root

The Polyakov Action

- Introduce metric on the world-sheet $h_{\alpha\beta}(\sigma, \tau)$

$$S_P = -\frac{T}{2} \int \mathrm{d}^2\sigma \, \sqrt{h} h^{\alpha\beta} \Gamma_{\alpha\beta}$$

using the definition: $h \equiv -\det h_{\alpha\beta}$

- Energy-momentum tensor defined as the response of S to varying $h_{\alpha\beta}$:

$$T_{\alpha\beta} := -\frac{1}{T} \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}}$$

it follows

$$T_{\alpha\beta} = \frac{1}{2} \Gamma_{\alpha\beta} - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \Gamma_{\gamma\delta}$$

- Energy-momentum conservation $\nabla^\alpha T_{\alpha\beta} = 0$

- Equation of motion for $h_{\alpha\beta}$:

$$\frac{\delta S}{\delta h^{\alpha\beta}} = 0$$

$$\boxed{T_{\alpha\beta} = 0}$$

$h_{\alpha\beta}$ is a non-propagating field $\longrightarrow$ auxiliary field

- Equation of motion for X^μ:

$$\frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta X^\mu) = 0$$

- If $h_{\alpha\beta}$ fulfills its equation of motion, then S_P and S_{NG} are classically equivalent:

$$\begin{aligned}
T_{\alpha\beta} &= \frac{1}{2}\Gamma_{\alpha\beta} - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\Gamma_{\gamma\delta} = 0 \quad |\ \det, \sqrt{} \\
\frac{1}{2}h^{\gamma\delta}\Gamma_{\gamma\delta} &= \frac{1}{\sqrt{h}}\sqrt{-\det\,\Gamma_{\alpha\beta}} \\
S_P &= -\frac{T}{2}\int \mathrm{d}^2\sigma\ \sqrt{h}h^{\alpha\beta}\Gamma_{\alpha\beta} \\
&= -T\int \mathrm{d}^2\sigma\ \sqrt{-\det\,\Gamma_{\alpha\beta}} = S_{NG}
\end{aligned}$$

- Symmetries of the Polyakov action
  - Poincaré invariance

$$\begin{aligned}
X^\mu &\to X^\mu + a^\mu{}_\nu X^\nu + b^\mu \\
\delta h_{\alpha\beta} &= 0
\end{aligned}$$

  - reparametrization invariance

$$\begin{aligned}
\sigma^\alpha &\to \tilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha(\sigma,\tau) \\
X^\mu(\sigma,\tau) &\to X^\mu(\sigma,\tau) + \xi^\alpha(\sigma,\tau)\partial_\alpha X^\mu \\
h_{\alpha\beta}(\sigma,\tau) &\to \frac{\mathrm{D}\sigma^\gamma}{\mathrm{d}\tilde{\sigma}^\alpha}\frac{\mathrm{D}\sigma^\delta}{\mathrm{d}\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma,\tau) \\
\delta h_{\alpha\beta} &= \frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta}
\end{aligned}$$

  - Weyl rescaling

$$\begin{aligned}
h_{\alpha\beta}(\sigma,\tau) &\to \Omega^2(\sigma,\tau)h_{\alpha\beta}(\sigma,\tau) \\
\Omega^2(\sigma,\tau) &= e^{2\Lambda(\sigma,\tau)} \\
&\approx 1 + 2\Lambda(\sigma,\tau) \\
\delta h_{\alpha\beta} &= 2\Lambda(\sigma,\tau)h_{\alpha\beta} \\
\delta X^\mu &= 0
\end{aligned}$$

- Consequence of Weyl invariance: tracelessness

$$\begin{aligned} \delta S &= -T \int \mathrm{d}^2\sigma \, \sqrt{h} T_{\alpha\beta} \delta h^{\alpha\beta} \\ h^{\alpha\beta} T_{\alpha\beta} &= 0 \end{aligned}$$

(without using the equation of motion $T_{\alpha\beta} = 0$)

- Conformal gauge: use reparametrization invariance to get locally:

$$h_{\alpha\beta} = \Omega^2(\sigma, \tau) \eta_{\alpha\beta}$$

$\Omega^2(\sigma, \tau)$ can be set to 1 by Weyl rescaling:

$$\boxed{h_{\alpha\beta} = \eta_{\alpha\beta}}$$

- Note: Although reparametrization and Weyl invariance have been used, reparametrizations

$$\delta h_{\alpha\beta} = \frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta} \propto h_{\alpha\beta}$$

followed by Weyl rescaling and thus only affect X^μ, not the metric $h_{\alpha\beta} = \eta_{\alpha\beta}$.

- World-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$: (notation $\sigma^x = (\sigma^+, \sigma^-)$)

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$$

and the metric transforms as

$$\tilde{h}_{xy}(\sigma^+, \sigma^-) = \frac{\partial\sigma^\gamma}{\partial\sigma^x} \frac{\partial\sigma^\delta}{\partial\sigma^y} h_{\gamma\delta}(\sigma, \tau) = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}$$

- Polyakov action in conformal gauge: the world-sheet metric is $\eta_{\alpha\beta} \Rightarrow \sqrt{\eta} = 1$

$$S_P = \frac{T}{2} \int \mathrm{d}^2\sigma \; (\dot{X}^2 - X'^2)$$

the conjugate momentum is: $\Pi^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = T\dot{X}^\mu$
and the Poisson brackets are:

$$\begin{aligned} \{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0 \\ \{X^\mu(\sigma,\tau), T\dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned}$$

Vanishing of energy-momentum tensor is in conformal gauge equivalent to:

$$\frac{1}{2}(\dot{X} \pm X')^2 = 0$$

- Equation of motion: varying with respect to X^μ:

$$\begin{aligned} \delta S &= T \int \mathrm{d}^2\sigma \; (\partial_\sigma^2 - \partial_\tau^2) X^\mu \delta X_\mu - T \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \; X'_\mu \delta X^\mu \Big|_0^{\bar{\sigma}} \stackrel{!}{=} 0 \\ \Rightarrow \quad & (\partial_\sigma^2 - \partial_\tau^2) X^\mu = 4\partial_+\partial_- X^\mu = 0 \end{aligned}$$

with the conditions:

$$\begin{aligned} X^\mu(\sigma + 2\pi, \tau) &= X^\mu(\sigma,\tau) \quad \text{(closed string)} \\ X'_\mu\Big|_0^{\bar{\sigma}} &= 0 \qquad\qquad\;\; \text{(open string)} \end{aligned}$$

general solution: $X^\mu(\sigma^+,\sigma^-) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)$

$X^\mu_{R,L}$ describe "right"- (respectively "left"-) moving modes of the string.

- Oscillator expansion

closed string

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

X^μ are real functions, i.e. $(X^\mu)^\dagger = X^\mu \Rightarrow$
- x^μ and p^μ are real
- $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$ and $\bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^\dagger$

Compute the center of mass momentum:

$$P^\mu = \int_0^{2\pi} \mathrm{d}\sigma\ T\dot{X}^\mu(\sigma,\tau) = p^\mu$$

and the center of mass position:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\ X^\mu(\sigma,\tau=0) = x^\mu$$

The Hamiltonian is defined as:

$$\begin{aligned} H &= \int_0^{\bar{\sigma}} \mathrm{d}\sigma\ (\dot{X}\cdot\Pi - L) \\ &= T\int_0^{\bar{\sigma}} \mathrm{d}\sigma\ ((\partial_+ X)^2 + (\partial_- X)^2) \\ &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) \end{aligned}$$

Notation $\alpha_0^\mu = \bar{\alpha}_0^\mu \equiv \frac{1}{\sqrt{4\pi T}}p^\mu$

open string

$$X_R^\mu(\sigma,\tau) = x^\mu + \frac{1}{\pi T} p^\mu \tau + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)$$

as in the closed string case:

- $\alpha_0^\mu \equiv \frac{1}{\sqrt{\pi T}} p^\mu$
- x^μ, p^μ are real
- $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$

and by inserting X^μ:

$$H = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

- Constraint $T_{\alpha\beta} \stackrel{!}{=} 0$ in light-cone coordinates:

$$\begin{aligned}
\tilde{T}_{xy}(\sigma^+,\sigma^-) &= \frac{\partial \sigma^\gamma}{\partial \sigma^x} \frac{\partial \sigma^\delta}{\partial \sigma^y} T_{\gamma\delta}(\sigma,\tau) \\
T_{++} &= \frac{1}{2} \partial_+ X \cdot \partial_+ X \stackrel{!}{=} 0 \\
T_{--} &= \frac{1}{2} \partial_- X \cdot \partial_- X \stackrel{!}{=} 0 \\
T_{+-} &\equiv T_{-+} \equiv 0
\end{aligned}$$

Energy-momentum conservation $\nabla^\alpha T_{\alpha\beta} = 0$ in light-cone coordinates:

$$\begin{aligned}
0 &= \partial_- T_{++} + \partial_+ T_{-+} = \partial_- T_{++} \\
0 &= \partial_+ T_{--} + \partial_- T_{+-} = \partial_+ T_{--}
\end{aligned}$$

This gives:

$$\begin{aligned}
T_{++} &= T_{++}(\sigma^+) \\
T_{--} &= T_{--}(\sigma^-)
\end{aligned}$$

Implies set of infinitely many conserved charges L_f:

$$\partial_-(f(\sigma^+)T_{++}) = 0$$
$$L_f = 2T \int_0^{\bar{\sigma}} \mathrm{d}\sigma \; f(\sigma^+)T_{++}(\sigma^+)$$

and analogously for T_{--}.

- Virasoro constraints and Virasoro algebra

closed string

Choose $f_m(\sigma^\pm) = \exp(im\sigma^\pm) \Rightarrow$ charges $\hat{=}$ Fourier coefficients of T_{xx} at $\tau = 0$:

$$\begin{aligned} L_m &= 2T \int_0^{2\pi} \mathrm{d}\sigma \; e^{-im\sigma} T_{--} \\ &= T \int_0^{2\pi} \mathrm{d}\sigma \; e^{-im\sigma} (\partial_- X)^2 \\ &= \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n \end{aligned}$$

$$\begin{aligned} \bar{L}_m &= 2T \int_0^{\bar{\sigma}} \mathrm{d}\sigma \; e^{+im\sigma} T_{++} \\ &= \frac{1}{2} \sum_n \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n \end{aligned}$$

The charges satisfy the Virasoro algebra:

$$\begin{aligned} \{L_m, L_n\}_{\text{P.B.}} &= -i(m-n)L_{m+n} \\ \{\bar{L}_m, \bar{L}_n\}_{\text{P.B.}} &= -i(m-n)\bar{L}_{m+n} \\ \{\bar{L}_m, L_n\}_{\text{P.B.}} &= 0 \end{aligned}$$

Compare to Hamiltonian:

$$H = L_0 + \bar{L}_0$$

open string

Because left- and right-movers are not independent, define:

$$
\begin{aligned}
L_m &= 2T \int_0^{\pi} \mathrm{d}\sigma \, \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}\right) \\
&= \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n
\end{aligned}
$$

again they satisfy:

$$\{L_m, L_n\}_{\text{P.B.}} = -i(m-n) L_{m+n}$$

Compare to Hamiltonian:

$$H = L_0$$

3 The Quantized Bosonic String

- Dirac's correspondence principle

$$
\begin{aligned}
X^\mu &\rightarrow \hat{X}^\mu \\
\alpha_m^\mu &\rightarrow \hat{\alpha}_m^\mu \\
\{\ ,\ \}_{\text{P.B.}} &\rightarrow \frac{1}{i}[\ ,\ ]
\end{aligned}
$$

- By analogy to Poisson brackets: commutator relations

$$
\begin{aligned}
[X^\mu(\sigma,\tau), T\dot{X}^\nu(\sigma',\tau)] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') \\
[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= [\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)] = 0
\end{aligned}
$$

by inserting the oscillator expansions

$$
\begin{aligned}
[x^\mu, p^\nu] &= i\eta^{\mu\nu} \\
[\alpha_m^\mu, \alpha_n^\nu] &= [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{(m+n),0}\ \eta^{\mu\nu} \\
[\bar{\alpha}_m^\mu, \alpha_n^\nu] &= 0
\end{aligned}
$$

- Compare to harmonic oscillator
  - the hermiticity condition still holds
  - rescale α_m^μ for $m > 0$:

$$
a_m^\mu \equiv \frac{1}{\sqrt{m}}\alpha_m^\mu \quad (a_m^\mu)^\dagger \equiv \frac{1}{\sqrt{m}}\alpha_{-m}^\mu
$$

they fulfill harmonic oscillator commutation relations

$$
[a_m^\mu, (a_n^\nu)^\dagger] = \delta_{m,n}\eta^{\mu\nu}
$$

So α_m are lowering operators for $m > 0$ and raising operators for $m < 0$.

The corresponding number operator (for $m > 0$) is: $N_m = \alpha_{-m}\alpha_m$.

- Ghosts (negative norm states)
  Denote ground state by: $|0, p^\mu\rangle$ consider $(m > 0)$:

$$\begin{aligned} [\alpha^0_m, \alpha^0_{-m}] &= -m \\ \langle 0|[\alpha^0_m, \alpha^0_{-m}]|0\rangle &= \langle 0|\alpha^0_m \alpha^0_{-m}|0\rangle - \underbrace{\langle 0|\alpha^0_{-m}\alpha^0_m|0\rangle}_{=0} \\ &= -m\langle 0|0\rangle \end{aligned}$$

  so we have a negative norm state:

$$\begin{aligned} \langle 0|\alpha^0_m \alpha^0_{-m}|0\rangle &= \langle 0|(\alpha^0_{-m})^\dagger \alpha^0_{-m}|0\rangle \\ &= \parallel \alpha^0_{-m}|0\rangle \parallel^2 \\ &= -m\langle 0|0\rangle \; < 0 \end{aligned}$$

  However: one can prove that in $d = 26$ these states decouple from the physical Hilbert space.

- The Virasoro algebra and normal ordering
  Constraint $T_{\alpha\beta} = 0$ corresponds to vanishing of L_m:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$

  but α_m now operators, so order matters. Define:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n:$$

  Normal ordering: raising operators to the left and lowering operators to the right. Problem only arises for L_0. Define L_0:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n$$

Introduce normal ordering constant a in all formulas by replacing L_0 by $(L_0 - a)$.

Now: determine the algebra of the L_m's

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$$

c appears due to quantum effects and is called central charge. Here: $c = \eta^\mu_\mu = d$.

Classically all L_m must vanish, but:

$$\langle\phi|[L_m, L_{-m}]|\phi\rangle = \langle\phi|2mL_0|\phi\rangle + \frac{d}{12}m(m^2-1)\langle\phi|\phi\rangle$$

so we can not require $L_m|\phi\rangle = 0 \ \ \forall m$, but only

$$\begin{aligned} L_m|\text{phys}\rangle &= 0 \qquad m > 0 \\ (L_0 - a)|\text{phys}\rangle &= 0 \end{aligned}$$

This defines the physical states $|\text{phys}\rangle$.

Indeed L_m form closed subalgebra for $m > 0$.

One can show that $(L_0 - \bar{L}_0)$ generates σ translations, which do not affect the string, so:

$$(L_0 - \bar{L}_0)|\text{phys}\rangle = 0$$

- Mass-shell condition

open string

Using the definitions $\frac{1}{2}\alpha_0^2 = \frac{1}{2\pi T} p^\mu p_\mu = \alpha' p^\mu p_\mu$ and $N = \sum_{n=1}^{\infty} N_m$ yields:

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \\ &= \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty} N_m \\ &= -\alpha' m^2 + N \end{aligned}$$

Applied on a physical state:

$$\begin{aligned} (L_0 - a)|\text{phys}\rangle &= 0 \\ \alpha' m^2 &= N - a \end{aligned}$$

closed string

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{\alpha'}{4} p^\mu p_\mu + N \\ \bar{L}_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n = \frac{\alpha'}{4} p^\mu p_\mu + \bar{N} \end{aligned}$$

the mass m is now given as:

$$m^2 = -p^\mu p_\mu = m_L^2 + m_R^2$$

L_0 and $\bar{L}_0$ applied on a physical state:

$$\begin{aligned} \alpha' m_L^2 &= 2(\bar{N} - a) \\ \alpha' m_R^2 &= 2(N - a) \end{aligned}$$

with $m_L^2 = m_R^2$ as a consequence of $L_0 - \bar{L}_0 = 0$.

- Light cone gauge

  Define: light cone coordinates in space-time

  $$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{d-1}) \qquad X^- = \frac{1}{\sqrt{2}}(X^0 - X^{d-1})$$

  New coordinates: $X^+, X^-, X^i,\ i = 1, \ldots, d-2$.

  Remaining reparametrization invariance:

  $$\frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta} \propto h_{\alpha\beta}$$

  in world-sheet light-cone coordinates:

  $$\partial_+\xi^- = \partial_-\xi^+ = 0 \ \text{ i.e. } \ \xi^\pm = \xi^\pm(\sigma^\pm)$$

  $\tilde{\sigma}^\pm = \sigma^\pm + \xi^\pm(\sigma^\pm)$ is the infinitesimal form of the reparametrization $\sigma^\pm \longrightarrow \tilde{\sigma}^\pm = \tilde{\sigma}^\pm(\sigma^\pm)$.

  Therefore

  $$\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$$

  transforms into

  $$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-))$$

  so $\tilde{\tau}$ arbitrary solution of wave equation:

  $$\partial_+\partial_-\tilde{\tau} = 0$$

  Choose $\tilde{\tau} \propto X^+$ and insert this into constraint

  $$\frac{1}{2}(\dot{X} \pm X')^2 = 0$$

  solve result with respect to $X^- = X^-(X^i)$ and thus X^+ and X^- are eliminated. One can show that

  $$\alpha_{-n} \cdot \alpha_n \Rightarrow \sum_{i=1}^{d-2} \alpha^i_{-n}\alpha^i_n$$

- Spectrum of the bosonic string
  States are generated by transverse oscillators acting on the ground state.

  open string spectrum

  Mass operator:

  $$\alpha' m^2 = (N - a)$$

  acting on ground state gives

  $$\alpha' m^2 |0, p^i\rangle = -a|0, p^i\rangle$$

  acting on first excited state gives

  $$\alpha' m^2 (\alpha^i_{-1}|0, p^j\rangle) = (1 - a)\alpha^i_{-1}|0, p^j\rangle$$

  $\alpha^i_{-1}|0, p^j\rangle$ is a vector of SO$(d-2)$. Lorentz group is SO(d).
  Lorentz invariance: little group is

  – SO$(d-2)$ for massless particles

  – SO$(d-1)$ for massive particles

  can not combine the fundamental SO$(d-2)$ representation to SO$(d-1)$. So first excited state must be massless.

  $$\text{Lorentz invariance} \;\Rightarrow\; a = 1$$

Normal ordering constant is fixed: $a = 1$. Consider

$$\begin{aligned} \frac{1}{2}\sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n &= \frac{1}{2}\sum_{n\neq 0} :\alpha^i_{-n}\alpha^i_n: + \underbrace{\frac{d-2}{2}\sum_{n=1}^{\infty} n}_{=-a} \\ &= \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \frac{d-2}{2}\sum_{n=1}^{\infty} n \end{aligned}$$

Consider: $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$ using the Riemann zeta function. Converges for $s > 1$, unique analytic continuation: $\zeta(-1) = -1/12$. It follows

$$\begin{aligned} -1 &= -\frac{d-2}{2}\frac{1}{12} \\ d &= 26 \end{aligned}$$

Due to Lorentz invariance $a = 1$ and $d = 26$

closed string spectrum

Excitation level for left- and right movers equal
Mass operator:

$$\alpha' m^2 = 4(N - a)$$

again $a = 1$ and $d = 26$

Table 3.1: The five lowest mass levels of the oriented open bosonic string

level	$\alpha'(\text{mass})^2$	states and their $SO(24)$ representation contents	little group	representation contents with respect to the little group
0	-1	$\vert 0\rangle$ $\bullet$ (1)	$SO(25)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\vert 0\rangle$ $\square$ (24)	$SO(24)$	$\square$ (24)
2	$+1$	$\alpha^i_{-2}\vert 0\rangle$ $\alpha^i_{-1}\alpha^j_{-1}\vert 0\rangle$ $\square$ $\square\square+\bullet$ (24) (299) + (1)	$SO(25)$	$\square\square$ (324)
3	$+2$	$\alpha^i_{-3}\vert 0\rangle$ $\alpha^i_{-2}\alpha^j_{-1}\vert 0\rangle$ $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\vert 0\rangle$ $\square$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square\square+\bullet$ $\square\square\square+\square$ (24) (276) + (299) + (1) (2576) + (24)	$SO(25)$	$\square\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (2900) + (300)
4	$+3$	$\alpha^i_{-4}\vert 0\rangle$ $\alpha^i_{-3}\alpha^j_{-1}\vert 0\rangle$ $\alpha^i_{-2}\alpha^j_{-2}\vert 0\rangle$ $\square$ $\square\square+\begin{smallmatrix}\square\\\square\end{smallmatrix}+\bullet$ $\square\square+\bullet$ (24) (299) + (276) + (1) (299) + (1) $\alpha^i_{-2}\alpha^j_{-1}\alpha^k_{-1}\vert 0\rangle$ $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\alpha^l_{-1}\vert 0\rangle$ $2\times\square+\square\square\square+\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$ $\square\square\square\square+\square\square+\bullet$ 2 × (24) + (2576) + (4576) (17250) + (299) + (1)	$SO(25)$	$\square\square\square\square + \begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$ (20150) + (5175) $+ \square\square + \bullet$ + (324) + (1)

Table 3.2: The three lowest mass levels of the oriented closed bosonic string

level	$\alpha'(\text{mass})^2$	states and their $SO(24)$ representation contents	little group	representation contents with respect to the little group
0	-4	$\lvert 0\rangle$ $\bullet$ (1)	$SO(25)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\bar{\alpha}^j_{-1}\lvert 0\rangle$ $\square \times \square$ (24) (24)	$SO(24)$	$\square\square \times \begin{smallmatrix}\square\\ \square\end{smallmatrix} \times \bullet$ (299) (276) (1)
2	$+4$	$\alpha^i_{-2}\bar{\alpha}^j_{-2}\lvert 0\rangle$: $\square \times \square$, (24) (24) $\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-1}\bar{\alpha}^l_{-1}\lvert 0\rangle$: $(\square\square + \bullet) \times (\square\square + \bullet)$, (299) + (1) (299) + (1) $\alpha^i_{-2}\bar{\alpha}^j_{-1}\bar{\alpha}^k_{-1}\lvert 0\rangle$: $\square \times (\square\square + \bullet)$, (24) (299) + (1) $\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-2}\lvert 0\rangle$: $(\square\square + \bullet) \times \square$, (299) + (1) (24)	$SO(25)$	$\square\square \times \square\square = \square\square\square\square + \begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}$ (324) (324) (20150) (32175) $+ \begin{smallmatrix}\square\square\square\square\\ \square\end{smallmatrix} + \square\square + \begin{smallmatrix}\square\\ \square\end{smallmatrix} + \bullet$ (52026) (324) (300) (1)

4 Summary

- Starting point: relativistic particle
- In analogy: string-action $\hat{=}$ area of world-sheet
- Equivalent: Polyakov action + constraints
- Quantization
- Constraints correspond to Virasoro algebra
- Zero mode Virasoro operator leads to mass^2
- Physical states organize into representations of little group
- Spin 2 (graviton), "spin 1" (antisymmetric tensor field) and spin 0 (dilaton) in spectrum of the <u>closed</u> string
- Problem: Tachyon with negative mass^2 $\Rightarrow$ Supersymmetric string theory