Physikalisches Institut
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Theoretische Physik

Exercise 9
Jan 6,8 2003
WS 03/04

Exercises on Elementary Particle Physics

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1. *Symmetry Breaking and Branching Rules*

   The basic idea of Grand Unification is that the Standard Model can be embedded in a simple group, e.g. SU(5). From a mathematical point of view, the problem is to determine the subalgebras of the simple group under consideration.

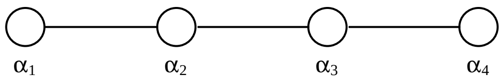


Figure 1: Dynkin diagram of SU(5)

   Dynkin's Symmetry Breaking Rule I

   Deleting any node with Kac-label $a_i = 1$ from the Dynkin diagram of an algebra gives a maximal regular subalgebra plus a U(1) factor. (The coefficients a_i of the decomposition of the highest root in the basis of simple roots are called Kac-labels.)

   (a) In the case of SU(5), all Kac-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the Standard Model, i.e.

   $$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

   (b) The lowest dimensional irrep of SU(5) is given by the highest weight $\theta = (1\,0\,0\,0)$. Using Dynkin's algorithm, calculate the weights in this irrep.

   (c) The irreducible representation of SU(5) corresponding to $\theta = (1\,0\,0\,0)$ is a reducible representation of SU(3) $\times$ SU(2). In exercise (a) you have learned that α_1, α_2 correspond to SU(3) and α_4 corresponds to SU(2). Consequently, every weight λ decomposes into

   $$\lambda = (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4) \to (\lambda_1\ \lambda_2 \mid \lambda_4) = (\mu|\nu).$$

   As a first step, write down all the weights $(\mu|\nu)$. Next, find the highest weight μ. Determine the weights and the dimension of the corresponding irrep. Consider now the values of ν belonging to this μ-irrep. What is the dimension of the ν-irrep? Repeat these steps starting with the ν-highest weight. The result reads

   $$\mathbf{5} \to (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}).$$

(d) Repeat the analysis for the irrep with highest weight $\theta = (0\,1\,0\,0)$. The result reads

$$\mathbf{10} \to (\mathbf{1}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}).$$

Hint: The highest weight of $\mathbf{3}$ is $(1\,0)$, and that of $\bar{\mathbf{3}}$ is $(0\,1)$. All weights which appear in the calculation have multiplicity 1.

(e) Repeat the analysis for the irrep with highest weight $\theta = (1\,0\,0\,1)$. The result reads

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\bar{\mathbf{3}}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}).$$

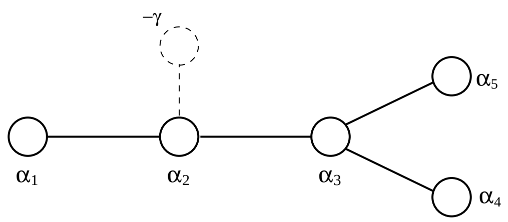
Hint: The highest weight of $\mathbf{8}$ is $(1\,1)$. All weights which appear in the calculation have multiplicity 1, except for $(0\,0\,0\,0)$ in $\mathbf{24}$ of SU(5), which has multiplicity 4, and $(0\,0)$ in $\mathbf{8}$ of SU(3), which has multiplicity 2.

The following exercise serves the purpose to illustrate Dynkin's second rule for symmetry breaking. It has no relevance for the following exercises or the lecture, and may thus be skipped.

Dynkin's Symmetry Breaking Rule II

Deleting any node whose Kac-label is a prime number from the extended Dynkin diagram of an algebra gives a maximal regular subalgebra, and the converse is also true, i.e. any maximal regular semi-simple subalgebra can be obtained by deleting an appropriate node.

(f) Consider the extended Dynkin diagram of SO(10):



All Kac-labels of SO(10) are prime. Apply Dynkin's rule to show that one possible symmetry breaking pattern is

$$\mathrm{SO}(10) \to \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2).$$

2. *The Standard Model in SU(5)*

(a) To recapitulate, fill in the transformation properties of the Standard Model fields (one generation of quarks and leptons) in the table below. The entry $(\mathbf{1},\mathbf{2})$, for example, denotes a singlet under $SU(3)_C$ and a doublet under $SU(2)_L$. As we want to deal with fields of one chirality only, we do not write down right-handed fields ψ_R, but instead their charge conjugates which transform as left-handed fields: $(\psi_R)^c = (\psi^c)_L =: \psi^c_L$.

Standard Model Particle Content			
$\ell_L = (\nu_e, e_L)$	**(1,2)**	e^c_L	
$q_L = (u_L, d_L)$		u^c_L	
W_μ		d^c_L	
B_μ		G_μ	

(b) To accomodate the particle content of the Standard Model in the $SU(5)$ GUT, we need the adjoint representation $\mathbf{24}$ and the two smallest representations, $\bar{\mathbf{5}}$ and $\mathbf{10}$. Associate the fields with the available representations (cf. Ex. 1).

$SU(3) \times SU(2)$ reps from $SU(5)$ $\bar{\mathbf{5}}$, $\mathbf{10}$ and $\mathbf{24}$.			
$(\bar{\mathbf{3}}, \mathbf{1})$		$(\mathbf{1}, \bar{\mathbf{2}})$	
$(\bar{\mathbf{3}}, \mathbf{1})$		$(\mathbf{3}, \mathbf{2})$	
$(\mathbf{1}, \mathbf{1})$		$(\mathbf{8}, \mathbf{1})$	
$(\mathbf{1}, \mathbf{3})$		$(\mathbf{1}, \mathbf{1})$	
$(\mathbf{3}, \mathbf{2})$		$(\bar{\mathbf{3}}, \mathbf{2})$	

(c) Do you need all representations to accomodate the Standard Model particles? Which additional particles do you get? Is there a representation suitable for a right-handed neutrino?

3. *Spontaneous Symmetry Breaking in $SU(5)$*

We want to describe the $SU(5)$ breaking by introducing a Higgs field in the adjoint, denoted as a 5×5 hermitean traceless matrix. (Note that this is not the Higgs field of the Standard Model, which would reside in the $\bar{\mathbf{5}}$.) Then we can write the scalar potential in the form

$$V(H) = -m^2 \mathrm{Tr}\left(H^2)\right) + \lambda_1 \left(\mathrm{Tr}\left(H^2\right)\right)^2 + \lambda_2 \mathrm{Tr}\left(H^4)\right),$$

where we have imposed a symmetry $H \to -H$ to remove a cubic term for simplicity.

(a) Show that H can be transformed into a real diagonal traceless matrix

$$H = U H_d U^\dagger \qquad \text{with} \quad H_d := \mathrm{diag}(h_1, h_2, h_3, h_4, h_5).$$

Hint: Use the $SU(5)$ transformation property $H \to H' = U H U^\dagger$

(b) Find that at the minimum of the scalar potential all h_i s satisfy the same cubic equation.

$$4\lambda_2 x^3 + 4\lambda_1 a x - 2m^2 x - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2.$$

where μ is a Lagrange multiplier which accounts for the constraint $\sum_i h_i = 0$.

This means that the vacuum expectation values of the h_i's can at most take on three different values ϕ_1, ϕ_2, ϕ_3. Let n_1, n_2, n_3 be the number of times ϕ_1, ϕ_2, ϕ_3 appear in $\langle H_d \rangle$:

$$\langle H_d \rangle := \mathrm{diag}(\phi_1, .., \phi_2, .., \phi_3) \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0.$$

This implies that the most general form of symmetry breaking is

$$SU(5) \to SU(n_1) \times SU(n_2) \times SU(n_3)$$

as well as additional $U(1)$ factors which leave $\langle H_d \rangle$ invariant. It turns out that, depending on the relative magnitude of the parameters λ_1 and λ_2, the combinations $(3, 2, 0)$ or $(4, 1, 0)$ for (n_1, n_2, n_3) minimize the potential. Thus,

$$SU(5) \to SU(3) \times SU(2) \times U(1)$$

or

$$SU(5) \to SU(4) \times U(1),$$

which would give restrictions on phenomenologically desireable values of λ_1, λ_2. (The details of the calculation can be found in L.-F. Li, Phys. Rev. D **9**, 1723 (1974).)