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Exercise 11
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Exercises on Elementary Particle Physics

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1. *β-Function of φ^4 Theory*

   In Exercise 10, (1.f), you learned that the running of the coupling constant λ is described by the equation

   $$t\frac{\partial\lambda(t)}{\partial t} = \beta(\lambda), \tag{1}$$

   i.e. if we know the β-function and the value of the coupling constant λ at one point t_0, we can calculate its value at another energy scale t. In Exercise 10, (1.b), you derived

   $$\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}$$

   for the β-function in φ^4 theory, where λ is the renormalized coupling constant, and μ is the renormalization scale introduced by dimensional regularization.

   (a) Using the relation between the bare and renormalized coupling constants in φ^4 theory [cf. Exercise 7, (1.a)]

   $$\delta_\lambda = \lambda_0 Z^2 - \lambda,$$

   show that the β-function is given by

   $$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}.$$

   (b) Inserting your result for the β-function into eq. (1), show that the running of λ is given by

   $$\lambda(t) = \frac{\lambda_0}{1 - \frac{3}{16\pi^2}\lambda_0 \log(t/t_0)}.$$

   Hint: Substitute $s = \log t$ in eq. (1). Then solve the differential equation.

2. *Beta Function and Anomalous Dimension in SU(N) Gauge Theory at One-Loop*

   In an SU(N) gauge theory the coupling is given by (take for example QCD from the lecture)

   $$\alpha(q^2) = \alpha_0 - \frac{\alpha_0^2\, b_0}{4\pi}\log\left(\frac{Q^2}{\mu^2}\right) + O(\alpha^3), \qquad Q^2 = -q^2.$$

(a) Beta function: Calculate the beta function at one-loop order. You will find

$$\beta(\alpha) = \frac{\partial \alpha}{\partial t} = -\frac{b_0}{2\pi}\,\alpha^2 + O(\alpha^3), \qquad t := \frac{1}{2}\log\left(\frac{Q^2}{\mu^2}\right).$$

Neglect the terms $O(\alpha^3)$, solve the differential equation and give the result for the running coupling.

(b) Anomalous Dimension: Integrate the equation for the anomalous dimension $\gamma(\alpha)$ from μ^2 to Q^2. Use that in an $SU(N)$ gauge theory at one-loop it is valid that

$$\gamma(\alpha) = -\frac{\gamma_0}{2\pi}\,\alpha, \qquad \gamma_0 = \frac{3}{2}\frac{N^2-1}{N}$$

to derive the mass running at one-loop

$$m(Q^2) = m(\mu^2)\left(\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right)^{\gamma_0/b_0}.$$

3. *Behaviour of* λ *Near a Simple Fixed Point*

In this exercise we derive the ultraviolet behaviour of λ in the case that the β function is given by

$$\beta(\lambda) = \lambda(a^2 - \lambda^2)$$

where a is a known constant. This example illustrates the typical behaviour of the running coupling near a simple fixed point.

(a) Start with the equation for the β function $\beta(\lambda) = t\,\partial\lambda(t)/\partial t$ and reparametrize it in terms of $u = \ln t$.

(b) The initial condition for the running coupling constant is $\lambda \to \lambda_0$ at $u = 0$. What is your expectation for $\lambda(u)$ for $u \to \infty$ if $\lambda_0 > 0$ or $\lambda_0 < 0$, respectively?

(c) Verify this by direct calculation: Choose $\lambda \in [-a, a]$. Integrate the β function and use the initial condition to arrive at

$$\frac{1}{2a^2}\left[\ln\frac{\lambda^2}{\lambda^2 - a^2} - \ln\frac{\lambda_0^2}{\lambda_0^2 - a^2}\right] = u.$$

Now solve for λ

$$\lambda = \frac{\pm\, a}{[1 - A\exp(-2a^2 u)]^2}, \qquad A = \frac{\lambda_0^2 - a^2}{\lambda_0^2}.$$

Show that for $\lambda > 0$ you have to choose the plus sign in order to agree with the initial condition. In the same way, show that for $\lambda < 0$ you have to choose the minus sign. Take the limit $u \to \infty$ and compare with your expectations.