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Exercise 2
31. Okt. 2005
WS 05/06

Exercises on Elementary Particle Physics

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1. *The adjoint Dirac equation and currents*

   (a) Define $\bar{\psi} = \psi^\dagger \gamma^0$ and use the covariant form of the Dirac equation to derive the adjoint Dirac equation
   $$i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0 \ .$$

   (b) Show that the probability current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is conserved. What can you say about the probability density j^0?

2. *Completeness relations*

   In exercise 1 we have seen that the solutions $u^{(1,2)}(p)e^{-ip\cdot x}$ describe free particles (e.g. an electron) of energy E and momentum $\vec{p}$, whereas the two negative energy solutions are to be associated with the antiparticles. We want to use the so called antiparticle description, namely that an antiparticle of energy E and momentum $\vec{p}$ is descibed by a $-E, -\vec{p}$ particle solution. For convenience define
   $$u^{(3,4)}(-p)e^{-i(-p)\cdot x} \equiv v^{(2,1)}(p)e^{ip\cdot x} \ .$$

   Note that then for the antiparticle $p^0 \equiv E \geq 0$! The v's are called antiparticle (e.g. positron) spinors.

   (a) Define $\not{p} \equiv \gamma^\mu p_\mu$ (we will use this 'slash' abbreviation for any four-vectors in the future). In exercise 1 we found
   $$(\not{p} - m)u(p) = 0 \ .$$
   What is the equivalent equation for $v(p)$?

   (b) What are the corresponding equations for $\bar{u}$ and $\bar{v}$?

   (c) Show that
   $$u^{(r)\dagger}u^{(s)} = 2E\delta_{rs} \ , \quad v^{(r)\dagger}v^{(s)} = 2E\delta_{rs} \ ,$$
   where r and s are running from 1 to 2 now of course.

   (d) Show that (no sum over (s) here)
   $$\bar{u}^{(s)}u^{(s)} = 2m \ , \quad \bar{v}^{(s)}v^{(s)} = -2m \ .$$

(e) Derive the completeness relations

$$\sum_{s=1,2} u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + m \,,$$
$$\sum_{s=1,2} v^{(s)}(p)\, \bar{v}^{(s)}(p) = \not{p} - m \,.$$

3. *Trace Theorems and Properties of γ-Matrices*

Proove the following equations without explicitly calculating a matrix product (i.e. only by using the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$):

$$\mathrm{Tr}\mathbb{1} = 4 \tag{1}$$
$$\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b \tag{2}$$
$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)] \tag{3}$$
$$\gamma_\mu \gamma^\mu = 4\mathbb{1} \tag{4}$$
$$\gamma_\mu \not{a} \gamma^\mu = -2\not{a} \tag{5}$$
$$\gamma_\mu \not{a}\not{b} \gamma^\mu = 4(a \cdot b)\mathbb{1} \tag{6}$$
$$\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a} \tag{7}$$