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Exercise 9
19. Dez. 2005
WS 05/06

Exercises on Elementary Particle Physics

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1. *Dynkin Diagram of SU(n) - part II*

   On the last exercise sheet, we discussed Dynkin diagrams of SU(n) Lie algebras in general. Then we followed the steps at the easiest example SU(2). Now, we want to consider a non-trivial example, i.e. SU(3).

   The standard basis for the hermitian 3×3 matrices are the so called Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

   (a) Define the step operators

$$T_\pm = \frac{1}{2}\left(\lambda_1 \pm i\lambda_2\right), \quad V_\pm = \frac{1}{2}\left(\lambda_4 \pm i\lambda_5\right), \quad U_\pm = \frac{1}{2}\left(\lambda_6 \pm i\lambda_7\right).$$

   Write them in terms of the e_{ab}'s.

   (b) Show that $H_1 = \frac{1}{2}\lambda_3$ and $H_2 = \frac{1}{2}\lambda_8$ are the generators of the Cartan subalgebra. Write them in terms of the e_{ab}'s and determine the coefficients λ_i (not to be confused with the Gell-Mann matrices).

   (c) Determine the action of the roots $\alpha_{e_{ab}}$ on the elements of the Cartan subalgebra H_1 and H_2 by using the results of Ex.8.2(c).

   New: Draw the roots in a 2-dim. picture, where the x-axis corresponds to H_1 and the y-axis to H_2, respectively.

(d) Define a basis of the Cartan subalgebra $\alpha_1 := \alpha_{e_{12}}$ and $\alpha_2 := \alpha_{e_{23}}$. They are simple by definition.

What are dual generators $H_{\alpha_i} \in H$ of the roots $\alpha_i \in H^*$ $(i = 1, 2)$ so that

$$\alpha_i(h) = \mathcal{K}(H_{\alpha_i}, h) \qquad \forall\, h \in H?$$

(e) Calculate the Cartan matrix and draw the Dynkin diagram of SU(3) by using the Killing form and the results of part (d).

New: In the case that the generators are normalized properly, you can also see $\langle \alpha_i, \alpha_j \rangle$ as a normal scalar product of the roots you have drawn in part (c).

2. *Representations of SU(n) - part I*

(a) Remember the definition of the adjoint ad $a(b) = [a, b]$. Show that the adjoint is a representation of the Lie algebra:

$$\text{ad}([a, b]) = [\text{ad}\ a, \text{ad}\ b] \qquad \text{for } a, b \in \mathfrak{g}$$

Note 1: The bracket $[.,.]$ denotes on the left hand side the abstract Lie bracket, but on the right hand side the commutator.

Note 2: The adjoint representation ad of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping

$$\text{ad} : \mathfrak{g} \to \text{End}(V) \ ,$$

where V is equal to the Lie algebra itself, i.e. $V = \mathfrak{g}$. Compare to Ex.3.1.

So, when we computed the Dynkin diagram of SU(n), we implicitly used the adjoint representation of SU(n):

$$\text{ad}\ h(e_{ab}) = [h, e_{ab}] \tag{1}$$

Furthermore, we had the eigenvalue equation

$$\text{ad}\ h(e_{ab}) = \alpha_{e_{ab}}(h) e_{ab} \tag{2}$$

which defined the roots $\alpha_{e_{ab}}$.

This eigenvalue equation can now be generalized to non-adjoint representations ρ on some vector space V. Let ϕ^i be a basis of V. We denote the representations of the elements of the Cartan subalgebra $h \in H$ by $\rho(h)$ and the representations of the step operators e_α by $\rho(e_\alpha)$. Then the eigenvalue equation (2) reads:

$$\rho(h)\phi^i = M^i(h)\phi^i$$

Since the linear functions M^i act on elements $h \in H$ and give (real) numbers, they are elements of the dual space H^*. They are called **weights**. The corresponding vectors ϕ^i are called **weight vectors**.

Note that roots are the weights of the adjoint representation!

We know that the simple roots α_j span H^*, so it is possible to express the weights by simple roots

$$M^i = \sum c_{ij}\alpha_j \ ,$$

where the coefficients c_{ij} are in general non-integers.

A weight M^i is called **positive** ($M^i > 0$), if the first non-zero coefficient is positive. We write $M^i > M^j$, if $M^i - M^j > 0$.

A weight is called the **highest weight**, denoted by Λ, if $\Lambda > M^i$ for all $M^i \neq \Lambda$.

(b) Suppose that ϕ^i is a weight vector with weight M^i. Show that $\rho(e_\alpha)\phi^i$ is a weight vector with weight $M^i + \alpha$ unless $\rho(e_\alpha)\phi^i = 0$.
Hint: Use the equations (1) and (2) and the fact that ρ is a representation.
Thus it makes sense to think of the $\rho(e_\alpha)$ as raising operators and the $\rho(e_{-\alpha})$ as lowering operators.

SU(3) example

We want to get used to the new ideas by considering the three dimensional representation of SU(3), denoted by **3**. For this representation, we know the explicit form of the SU(3) generators as 3×3 matrices. They are listed in Ex.9.1. The Cartan subalgebra H is spanned by the elements H_1 and H_2. A basis of the three dimensional vector space is given by the weight vectors

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \phi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Consider the action of $H = aH_1 + bH_2$ on the weight vectors ϕ^i to find the weights M^i. Express the weights M^i in terms of the simple roots α_1 and α_2 and find the highest weight Λ. Hint: Use the results of Ex. 9.1(c).

(d) Along the lines of Ex.9.1(c), draw the weights in a 2-dim. picture. Note that the difference of two weights is a root!

(e) Consider the action of $T_\pm$, $V_\pm$ and $U_\pm$ on the weight vectors ϕ^i and compare your results to part (b).
Indicate the action of these operators on the weights M^i by arrows in the picture of part (d).

Without any prove: A highest weight Λ can be specified by a set of non-negative integers, called the **Dynkin coefficients**:

$$\Lambda_i = 2\frac{\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

If one only has the Dynkin coefficients of the highest weight of some algebra, it is possible to determine the corresponding highest weight and, furthermore, one can even reconstruct all weights by the following recipe:

- To start, we need the Dynkin coefficients and the Cartan matrix.
- For each non-negative Dynkin coefficient Λ_i of a weight substract the i-th row of the Cartan matrix. You will get the Dynkin coefficients of another weight. You repeat substracting the i-th row of the Cartan matrix from the original weight in total Λ_i-times.
- Repeat the last step for all weights, until all Dynkin coefficients are non-positive.

The number of Dynkin coefficients gives the number of different weights and therefore the number of linear independent weight vectors. So this gives the dimension of the representation. (This is not always true. In the case of the adjoint representation the Dynkin coefficient (0...0) corresponds to the Cartan subalgebra, which can contain more than one linear independent weight (root) vector).

SU(3) example

(e) Compute the Dynkin coefficients of the three weights and check that the **highest weight construction** is correct.

(f) Perform the highest weight construction for the Dynkin coefficients $(1, 1)$ of SU(3).

More group theory in the book of Cahn: *http://www-physics.lbl.gov/~rncahn/book.html*

3. *The CKM Matrix*

Consider the Yukawa couplings of N generations of quarks after spontaneous symmetry breaking (and denote generation indices by i and j, sum over repeated indices).

$$\mathcal{L} \supset -G_d^{(ij)} \bar{d}_{Li} d_{Rj} - G_u^{(ij)} \bar{u}_{Li} u_{Rj} + h.c.$$

(a) Why "+*h.c.*"?

The real matrices G_d and G_u do not need to be diagonal. So, the quarks d and u are not mass eigenstates, but they are eigenstates of the weak interaction. Nevertheless, only mass eigenstates are regarded as physical particles that can be detected in an experiment. So we have to perform a basis transformation and diagonalize the mass matrices.

(b) Use biunitary transformations $S_d G_d T_d^\dagger = G_d^{\text{diag}}$ and $S_u G_u T_u^\dagger = G_u^{\text{diag}}$ to diagonalize the mass matrices ($S_{d/u}$ and $T_{d/u}$ are unitary matrices).

(c) Next, we analyse how this change of basis affects the weak interaction. The relevant term of the Lagrangian is (using $L_i = (u_{Li}, d_{Li})^T$):

$$\begin{aligned} \mathcal{L} &\supset \bar{L}_i i\gamma^\mu D_\mu L_i \\ &\supset -\frac{g}{\sqrt{2}} \left(W_\mu^+ \bar{u}_{Li} \gamma^\mu d_{Li} + W_\mu^- \bar{d}_{Li} \gamma^\mu u_{Li} \right) \end{aligned}$$

Use the mass eigenstates of the quarks to investigate the weak interaction vertex.