Universität Bonn
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Example sheet 6
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General Relativity

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1. *The Einstein-Hilbert action*

The field equations of General Relativity can be derived from a variational principle. Let's see how this works!

(a) Prove the four useful identities

$$\begin{array}{lll} (i) & \delta g^{\mu\nu} & = -g^{\mu\kappa} g^{\nu\lambda} \delta g_{\kappa\lambda}\,, \\ (ii) & \delta g & = g g^{\mu\nu} \delta g_{\mu\nu}\,, \\ (iii) & \delta\sqrt{-g} & = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}\,, \\ (iv) & \delta R_{\mu\nu} & = \mathrm{D}_\nu \delta \Gamma^\kappa{}_{\kappa\mu} - \mathrm{D}_\kappa \delta \Gamma^\kappa{}_{\mu\nu}\,, \end{array}$$

where $g \equiv \det g_{\mu\nu}$. *(Hint: To prove (ii) use* $\ln(\det g_{\mu\nu}) = \mathrm{tr}(\ln g_{\mu\nu})$*.)*

(b) Define the *Einstein-Hilbert* action by

$$S_{\mathrm{EH}} \equiv \frac{1}{16\pi G} \int R\sqrt{-g}\,\mathrm{d}^4 x\,, \tag{1}$$

where the constant factor $\frac{1}{16\pi G}$ is introduced to reproduce the Newtonian limit when matter is added. Show that under the variation $g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G} \int \left(-R^{\mu\nu} + \frac{1}{2} R\, g^{\mu\nu} \right) \delta g_{\mu\nu} \sqrt{-g}\,\mathrm{d}^4 x\,.$$

(c) Suppose there exists matter described by an action

$$S_{\mathrm{M}} \equiv \int \mathcal{L}\sqrt{-g}\,\mathrm{d}^4 x\,, \tag{2}$$

where $\mathcal{L}$ is the Lagrangian density of the theory. If the matter action changes by δS_{M} under $\delta g_{\mu\nu}$, the *energy-momentum tensor* $T^{\mu\nu}$ is defined by

$$\delta S_{\mathrm{M}} = \frac{1}{2} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{-g}\,\mathrm{d}^4 x\,. \tag{3}$$

What is the expression for $T^{\mu\nu}$ in terms of $\mathcal{L}$?

(d) Write down the field equations from varying $S_{\mathrm{EH}}+S_{\mathrm{M}}$ with respect to the metric.

(e) We may add an extra scalar to the scalar curvature without spoiling the invariance of the action. Add a constant Λ, called the *cosmological constant* and write down the modified field equations.

2. Schwarzschild solution (Episode I)

A general ansatz of writing a spherically symmetric and static line element is

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{4}$$

Traditionally, this has been called the *standard form*. We will now (and on the next exercise sheet) try to determine $A(r)$ and $B(r)$ from the Einstein equations.

(a) What are the non-zero components of the metric tensor?

(b) For all, who want to calculate some Christoffel symbols: Calculate all non-zero Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\,. \tag{5}$$

(You should find 8 distinct terms.)

(c) And for all, who still want to calculate more: Calculate the terms of the Ricci tensor

$$R_{\mu\kappa} \equiv \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}\,. \tag{6}$$

(Hint: $R_{\mu\nu} = 0$ for $\mu \neq \nu$.)

...and we will continue these nasty calculations next week...