Universität Bonn
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Theoretische Physik

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Example sheet 7
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General Relativity

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1. *Energy-Momentum in Hydrodynamics*

A comoving observer in a *perfect fluid* will by definition see his surroundings as isotropic. In this frame, the energy-momentum tensor will be

$$\tilde{T}^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \tag{1}$$

where ρ is the density and p the pressure of the liquid.

(a) Calculate the energy-momentum tensor $T^{\mu\nu}$ for the rest frame. Assume the comoving observer's velocity to be $\vec{v}$.

(b) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p + \rho)\, U^\mu U^\nu + p\, \eta^{\mu\nu} \tag{2}$$

where U^μ is the four-velocity of the fluid.

(c) Consider an ideal gas (point particles that only interact in local collisions). Its energy-momentum tensor is

$$T^{\mu\nu} = \sum_N \frac{p_N^\mu\, p_N^\nu}{E_N}\, \delta^3(\vec{x} - \vec{x}_N)\, . \tag{3}$$

Calculate the density ρ and pressure p for a comoving observer.

(d) What is the relation between ρ and p for a non-relativistic gas?
What is the relation for a highly relativistic gas?
(What relation exists between E and $\vec{p}$ in those limits?
Use the particle number density $n \equiv \sum_N \delta^3(\vec{x} - \vec{x}_N)$.)

2. Normal coordinates

In analogy to the Cartesian coordinates of the euclidean space, which are just the geodetic lines, we want to define the *Riemannian normal coordinate system* in a curved space.

(a) Expand the geodesic $x^i(s)$ in a small region around the point P_0 (with coordinates x_0).

(b) Define $\xi^i \equiv \left(\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)_0$ and use the geodesic equation to rewrite the expansion as

$$x^i(s) = x^i_o + \xi^i s - \frac{1}{2}\Gamma^i{}_{jk}\xi^j\xi^k s^2 - \frac{1}{3!}\Gamma^i{}_{klm}\xi^k\xi^l\xi^m s^3 \dots,$$

where the Γ's have to be taken at x_0.

(c) Define the Riemannian normal coordinates by $y^i \equiv \xi^i s$ and show that these new coordinates are just the geodesic lines.
Thus a geodesic through P_0 in Riemannian normal coordinates reads $y^i(s) = \xi^i s$.

(d) Show that in the normal coordinate system the Christoffel symbols vanish at P_0.

(e) Why does the expansion of the metric $\tilde{g}_{ik}$ around P_0 in normal coordinates reads

$$\tilde{g}_{ik} = \tilde{g}_{ik}(0) + c_{iklm}y^l y^m + \dots ?$$