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# General Relativity

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## 1. Gravitational waves

In order to describe gravitational waves, in the lecture the metric was decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (1)$$

with  $h_{\mu\nu} \ll 1$ , so that we could work in linear order in  $h$ . The Einstein field equations in free space in this *weak field approximation* read

$$\partial^\lambda \partial_\lambda h_{\mu\nu} - \partial_\mu \partial_\lambda h^\lambda{}_\nu - \partial_\nu \partial_\lambda h^\lambda{}_\mu + \partial_\mu \partial_\nu h^\lambda{}_\lambda = 0 . \quad (2)$$

(a) Show that the field equations in the *harmonic gauge*

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\mu{}_\mu \quad (3)$$

reduce to a wave equation.

(b) Make an ansatz for the solution of the field equation for gravitational waves

$$h_{\mu\nu}(x) = e_{\mu\nu} \exp(ik^\lambda x_\lambda) + e_{\mu\nu}^* \exp(-ik^\lambda x_\lambda) . \quad (4)$$

Show that  $h$  solves the field equations if

$$k^\mu k_\mu = 0 \quad (5)$$

and that the choice of a harmonic coordinate system Eq. (3) corresponds to

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu . \quad (6)$$

Why is the matrix  $e_{\mu\nu}$  symmetric?

(c) Consider a wave traveling in  $z$ -direction, i.e.

$$k^1 = k^2 = 0 \quad \text{and} \quad k^3 = k^0 =: k > 0 . \quad (7)$$

Express  $e_{i0}$  ( $1 \leq i \leq 3$ ) and  $e_{22}$  in terms of the other  $e_{\mu\nu}$ 's.

(d) How does  $h_{\mu\nu}$  change under a coordinate transformation

$$x^\mu \mapsto x'^\mu \equiv x^\mu + \varepsilon^\mu ? \quad (8)$$

Perform a coordinate transformation with

$$\varepsilon^\mu(x) = i\varepsilon^\mu \exp(ik^\lambda x_\lambda) - i\varepsilon^{\mu*} \exp(-ik^\lambda x_\lambda) \quad (9)$$

and determine the  $h_{\mu\nu}$ .

(e) Invent a coordinate transformation that brings all  $e_{\mu\nu}$  to 0 except for  $e_{11}$ ,  $e_{12}$  and  $e_{22}$ . How many physical components does  $h$  have?

\* **2. Robertson-Walker metric**

The *Robertson-Walker metric* reads

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - \alpha r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\} \quad (10)$$

with  $\alpha = 0, \pm 1$ . Compute

- (a) the Christoffel symbols.
- (b) the spatial Riemann tensor, the spatial Ricci tensor and the spatial curvature scalar.
- (c) the (4-dimensional) Riemann and Ricci tensors as well as the curvature scalar.

♡ The exam will be on Thursday, February 8<sup>th</sup>, 2007 from 14:15pm - 16:00pm in the lecture hall I.