General Relativity and Cosmology

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Example sheet 2

Electromagnetism

1. A typical example of a(n antisymmetric) tensor, is the *electromagnetic field strength* tensor $F_{\mu\nu}$.

   (a) Show that $\partial_{[\mu}F_{\lambda\nu]} = 0$ is equivalent to (here $[\dots]$ means antisymmetrisation, see handout 1)

   $$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

   (b) Write the *contravariant* components of $F^{\mu\nu}$ identifying $F^{0i} \equiv E^i$, $F^{ij} \equiv \epsilon^{ijk}B_k$ (where ϵ^{ijk} is the antisymmetric Levi-Civita symbol in 3 dimensions).

   (c) Show that Maxwell's equations can be recovered from the *equations of motion* for $F^{\mu\nu}$, given by

   $$\partial_\mu F^{\nu\mu} = J^\nu \,, \qquad \partial_{[\mu}F_{\lambda\nu]} = 0$$

   Identify the components of the four current J^μ.

   (d) The electric $\mathbf{E}$ and magnetic $\mathbf{B}$ vectors can be expressed in terms of a *vector potential* $\mathbf{A}$ and a *scalar potential* ϕ as

   $$\mathbf{B} = \nabla \times \mathbf{A} \,, \qquad \mathbf{E} = \nabla\phi - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}$$

   How is $F_{\mu\nu}$ related to $\mathbf{A}$ and ϕ? (write $\mathbf{A}$ and ϕ as a quadrivector A_μ).

   (e) The potentials $\mathbf{A}$ and ϕ discussed above, are not unique. We can replace

   $$\mathbf{A} \to \mathbf{A} + \nabla\psi \,, \qquad \phi \to \phi + \frac{\partial\psi}{\partial t}$$

   or equivalently, $A_\mu \to A_\mu + \partial_\mu\psi$. This is called a *gauge transformation* Show that $F_{\mu\nu}$ is indeed invariant under this gauge transformation

   (f) From the tensor Lorentz transformation rules for a tensor $F_{\mu\nu}$, show how $\mathbf{E}$ and $\mathbf{B}$ transform under

   i) a boost about the x axis.

   ii) a rotation about the x axis (*)

   (g) With the fields $\mathbf{E}$ and $\mathbf{B}$, we can form invariant quantities with respect to a transformation from one system of reference to another. These are

   $$F_{\mu\nu}F^{\mu\nu} \,, \qquad F_{\mu\nu}F_{\lambda\beta}\epsilon^{\mu\nu\lambda\beta}$$

   Show that they are indeed invariant.

   Write these two conditions above in terms of $\mathbf{E}$ and $\mathbf{B}$ and explain the physics involved.

   (h) Verify that (*)

   $$f^\mu \equiv \frac{dp^\mu}{d\tau} = e\,F^\mu{}_\nu\,\frac{d\,x_n^\nu(t)}{d\,\tau} \tag{1}$$

   is the correct equation for the electromagnetic four-force f^μ acting on a charged particle ($p^\mu = m dx^\mu/d\tau$) Taking the limit of small velocities, show that it does reproduces the Lorentz force.

Energy-momentum tensor (*)

2. It is possible to define a charge and current density for the four momentum p^μ, this is the *energy-momentum tensor*

$$T^{\mu\nu}(\mathbf{x},\, t) \;=\; \sum_n p_n^\mu(t)\, \frac{d\, x_n^\nu(t)}{d\, t}\, \delta^3(\mathbf{x} - \mathbf{x}_n(t))$$

   (a) Show that the energy-momentum tensor is only conserved up to a *force density* G^μ which vanishes for free particles

$$\partial_\nu T^{\mu\nu} \;=\; G^\nu$$

   (b) Check that for electromagnetic forces given in (1)

$$G^\mu \;=\; F^\mu_{\ \nu} J^\nu$$

   (c) To obtain a conserved energy-momentum tensor, we have to include the contribution of the electromagnetic field itself:

$$T^{\mu\nu}_{em} \;=\; F^\mu_{\ \rho}\, F^{\nu\rho} - \frac{1}{4}\, \eta^{\mu\nu} F^{\rho\sigma}_{\rho\sigma}$$

   Write T^{00}_{em} and T^{i0}_{em} in terms of the electric and magnetic vectors $\mathbf{E}$ and $\mathbf{B}$. Do you recognize the resulting expressions?

   (d) Show that $\partial_\nu T^{\mu nu}_{em}$ cancels G^μ introduced in point (a).

   (e) Show that the total momentum $p^\mu \;=\; \int\, d^3x\, T^{\mu 0}(\mathbf{x},\, t)$ is a conserved quantity.

Angular momentum (*)

3. Consider another conserved quantity M given by

$$M^{\rho\mu\nu} \;=\; x^\mu\, T^{\nu\rho} - x^\nu T^{\mu\rho} \tag{2}$$

   (a) Show that

$$J^{\mu\nu} \;=\; \int d^3x\, M^{0\mu\nu} \tag{3}$$

   is antosymmetric and can be interpreted as the angular momentum of the system.

   (b) How does $J^{\mu\nu}$ transform under $x^\mu \to x^\mu + a^\mu$?
   What is the physical interpretation of the extra terms?

   (c) Show that the quantity $S_\mu \;=\; \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, J^{\nu\rho}\, u^\sigma$ (where $u^\sigma \;=\; p^\sigma/\sqrt{-p\cdot p}$) is the system's four velocity) is invariant under the translation a^μ. What are the components of S in the center of mass frame of the system? What is the physical interpretation of S?