General Relativity and Cosmology

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Example sheet 4

1. **Non-orthogonal coordinates**

   Consider a Cartesian coordinate system spanned by the unit vectors $\vec{\epsilon_1}$ and $\vec{\epsilon_2}$, and a non-orthogonal system spanned by $\vec{e}_1$ and $\vec{e}_2$. The tilted system's basis vectors are given by

   $$\vec{e}_1 = \vec{\epsilon_1}\,, \qquad \vec{e}_2 = \vec{\epsilon_1} + 2\vec{\epsilon}_2\,.$$

   We can write any point X in the Cartesian system as (here ξ^a are the contravariant coordinates)

   $$X = \xi^1 \cdot \vec{\epsilon}_1 + \xi^2 \cdot \vec{\epsilon}_2 \equiv \xi^a \cdot \vec{\epsilon}_a \equiv \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}_\epsilon .$$

   (a) Rewrite X in the tilted system to obtain the contravariant coordinates x^i of $\vec{\xi}$ in terms of the ξ^a.

   (b) Show that the distance s from point X to the origin can be written in the Cartesian system as $s^2 = \eta_{ab}\,\xi^a\xi^b$. What does the metric η_{ab} of the Cartesian system look like?

   (c) If we require that distances should not depend on the choice of coordinate system, we can write s^2 as

   $$s^2 = \eta_{ab}\xi^a\xi^b = g_{ij}x^i x^i\,,$$

   where we now denote the metric of the tilted system by g_{ij}.
   Write g_{ij} in terms of η_{ab} for our example and generally $(dx^i = (\partial x^i/\partial\xi^a)d\xi^a)$.

   (d) Draw a sketch of the two coordinate systems and choose a point X arbitrarily. Identify the coordinates ξ^a, ξ_a, x^i and x_i of your chosen point on the axis.

2. **Locally inertial coordinates**

   The equivalence principle states that nearby any point $x_{(0)}$ in a curved space, it is always possible to find a system of *locally inertial coordinates* in which the space appears flat. The aim of this exercise is to show that this is indeed the case[1].

   (a) Convince yourself that this is equivalent to impose that, in the new coordinate system, the metric is such that

   $$g_{\hat{\mu}\hat{\nu}}\left(\hat{x}_{(0)}\right) = \eta_{\hat{\mu}\hat{\nu}} \qquad \partial_{\hat{\rho}}\, g_{\hat{\mu}\hat{\nu}}\left(\hat{x}_{(0)}\right) = 0 \tag{1}$$

   (b) Consider the transformation law for the metric:

   $$g_{\hat{\mu}\hat{\nu}}\left(\hat{x}_{(0)}\right) = \frac{\partial x^\rho}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial x^\lambda}{\partial \hat{x}^{\hat{\nu}}}\, g_{\lambda\rho}\left(x_{(0)}\right)$$

   Since we want to focus on the point $x_{(0)}$ (and its coordinate transformed $\hat{x}_{(0)}$) we can expand both sides of the previous relation in Taylor series around this point. Show that, stopping at the first term in the expansion, one can write (all functions are evaluated at $\hat{x}_{(0)}$):

   $$\begin{aligned} g_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}\left(\hat{x}^{\hat{\rho}} - \hat{x}^{\hat{\rho}}_{(0)}\right) + \ldots \;=\; & \frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}} g_{\mu\nu} \\ & + \left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial^2 x^\nu}{\partial \hat{x}^{\hat{\rho}}\partial \hat{x}^{\hat{\nu}}} g_{\mu\nu} + \frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}}\partial_{\hat{\rho}} g_{\mu\nu}\right)\left(\hat{x}^{\hat{\rho}} - \hat{x}^{\hat{\rho}}_{(0)}\right) + \ldots \end{aligned}$$

   (c) By carefully counting the free parameters contained in the matrices $\partial x/\partial\hat{x}$ and $\partial^2 x/\partial\hat{x}^2$, show then that condition (1) can be satisfied.

[1] Note that a very special and useful realisation of such a local inertial frame is the *Riemann normal coordinate system*.