General Relativity and Cosmology

Winter term 2008/09

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Example sheet 9

Problems marked with an asterisk are optional.

1. **Einstein Equations**

(a) Consider the Einstein-Hilbert action for gravity

$$S = \frac{1}{\kappa} \int d^4x \sqrt{-g}\, R.$$

Derive Einstein's equations of motion

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0\,,$$

by varying the action with respect to the metric. You need to compute three terms $\delta\sqrt{-g}$, $\delta g^{\mu\nu}$ and $\delta R_{\mu\nu}$.

(b) Concentrate on the variation $\delta R_{\mu\nu}$. Does this vanish in general? Explain.

(c) (*)The term you have computed above (b), needs to be compensated by a *boundary term*, some times referred to as *Gibbons-Hawing* term. The variation of the boundary terms compensates exactly the term you just calculated. Can you guess the form of such boundary term?

(d) Starting from

$$-g_{\nu\rho} \hat{D}_\lambda \left(\sqrt{-g} g^{\nu\rho}\right) = 0\,,$$

where $\hat{D}_\lambda$ is the covariant derivative with a general connection, show that the determinant of the metric, g, is covariantly constant w.r.t. $\hat{D}_\lambda$ (i.e. fill in the steps in the lecture calculations).

(e) Consider the following term in the gravitational action

$$\int d^4x\, a(x)\, R^{\mu}{}_{\nu\lambda\delta}\, R^{\nu}{}_{\mu\alpha\beta}\, \epsilon^{\lambda\delta\alpha\beta}\,.$$

Discuss the following points in as much detail as you can.

i. In order for this to be a valid term in the action, how should the *field* $a(x)$ transform ? (why is $\sqrt{-g}$ not present?)

ii. Why was this term not included in the Einstein-Hilbert action on the same footing as R ?

(f) (*) Argue that the action

$$\int d^4x \sqrt{\det(R_{(\mu\nu)}(\hat{\Gamma}))}\,,$$

is equivalent to the Einstein-Hilbert action with a nonzero cosmological constant.

2. **Energy-Momentum in Hydrodynamics**

A comoving observer in a *perfect fluid* will by definition see her surroundings as *isotropic*. In this frame, the energy-momentum tensor will be

$$\tilde{T}^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

where ρ is the energy density and p the pressure of the fluid.

(a) Calculate the energy-momentum tensor $T^{\mu\nu}$ for the rest frame. Assume the comoving observer's velocity to be $\vec{v}$.

(b) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p + \rho)U^\mu U^\nu + p\,\eta^{\mu\nu}$$

where U^μ is the four-velocity of the fluid.

(c) Consider an ideal gas (point particles that only interact in local collisions). Its energy momentum tensor is

$$T^{\mu\nu} = \sum_N \frac{p_N^\mu p_N^\nu}{E_N}\,\delta^3(\vec{x} - \vec{x}_N)$$

Calculate the density ρ and pressure for a comoving observer.

(d) What is the relation between ρ and p for a non-relativistic gas?
What is the relation for a highly relativistic gas?
(What relation exists between E and $\vec{p}$ in those limits? Use particle number density $n \equiv \sum_N \delta^3(\vec{x} - \vec{x}_N)$.)