String Theory

Winter Term 2008/2009

Problem Sheet 5

Discussion: November 26, 14:00 in Hörsaal 118, AVZ

1. For $SO(3)$, the generators $\mathcal{J}_a$ in the fundamental representation are given by

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \qquad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} .$$

The spin s of a multiplet Φ is determined by the eigenvalue of the Casimir operator $\mathcal{J}^2 = \sum_i \mathcal{J}_i^2$, where $\mathcal{J}^2\Phi = s(s+1)\Phi$.

(a) Show that the triplet representation $\Phi = \phi_i$ has spin one!

(b) Consider the two-index representation $\Phi = \phi_{ij}$. Show that the algebra acts on Φ as $\mathcal{J}_a\Phi = [J_a, \Phi]$!
Furthermore, show that Φ is not an irreducible representation by demonstrating that it does not have definite spin. Decompose Φ into its irreducible components and determine their spin.

2. In D dimensions, the Clifford algebra is given by D matrices Γ^μ which satisfy

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} .$$

Here, $\mu = 0, \dots, D-1$ and $\eta = \mathrm{diag}(-, +, \dots, +)$.

(a) Show that the matrices

$$\Sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\left[\Gamma^\mu, \Gamma^\nu\right]$$

form a representation of the Lorentz algebra (see previous problem sheet). This representation is called spinor representation, and the elements of the representation space are (Dirac) spinors

(b) Define a new matrix Γ_* by

$$\Gamma_* = \mathrm{i}^\alpha \Gamma^0 \cdots \Gamma^{D-1} .$$

α is a parameter to be determined later. Show that Γ_* (anti)commutes with the Γ^μ,

$$\{\Gamma_*, \Gamma^\mu\} = 0 \quad \text{for } D \text{ even} , \qquad [\Gamma_*, \Gamma^\mu] = 0 \quad \text{for } D \text{ odd.}$$

(Note that this implies that for odd D, Γ_* is a multiple of the unit matrix.)
Show that $\Gamma_*^2 \sim \mathbb{1}$, and find (for even D) an α such that $\Gamma_*^2 = \mathbb{1}$!

(c) For even D, define the operators $P_\pm = \frac{1}{2}(1 \pm \Gamma_*)$. Verify that they form a complete set of orthogonal projectors! These projectors define right- and left-chiral (Weyl) spinors.

Prove that the representation of the Lorentz group by the generators $\Sigma^{\mu\nu}$ is reducible. To do so, show that it splits into two mutually commuting representations generated by the chiral generators $\Sigma_+^{\mu\nu} = \Sigma^{\mu\nu} P_+$ and $\Sigma_-^{mu\nu}$!

3. Consider a spinor $\psi = (\psi_1, \psi_2)^T$ in $D = 2$. The Γ matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

4. Find the Lorentz generator Σ^{01}! Determine the action of the Lorentz group by $\exp\{\mathrm{i}\omega_{01}\Sigma^{01}\}$ on ψ, where ω_{01} is a real parameter. How does the Lorentz group act on the chiral components of the spinor?

5. A Majorana condition is a reality condition on the spinor of the form

$$\psi^* = B\psi$$

with some invertible matrix B. Show that consistency requires $BB^* = 1$ and $B\Sigma^{01}B^{-1} = -\Sigma^{01*}$!

Find a matrix B that works and show that it is compatible with chirality, i.e. that the reality condition can be imposed on the chiral components!

6. Let the supersymmetry generator be given by

$$Q_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} + \mathrm{i}\left(\gamma^a\theta\right)_\alpha \partial_a\,.$$

Here $a = 0, 1$ label the worldsheet coordinates.

(a) Verify the commutation relation

$$[\bar\epsilon_1 Q, \bar\epsilon_2 Q] = 2\mathrm{i}\bar\epsilon_1\gamma^a\epsilon_2\partial_a\,.$$

Here the ϵ_i are anticommuting (Majorana) parameters.

(b) Show that its action on superfield $Y(\sigma^a, \theta)$ is given by

$$e^{\bar\epsilon Q} Y(\sigma^a, \theta) = Y\left(\sigma^a + \mathrm{i}\bar\epsilon\gamma^a\theta + \frac{\mathrm{i}}{2}\bar\epsilon\gamma^a\epsilon, \theta + \epsilon\right)$$

(c) Comparing this with the expansion $Y = X + \bar\theta\psi + \frac{1}{2}\bar\theta\theta B$, find the off-shell SUSY transformations δX, $\delta\psi$ and δB!