String Theory

Winter Term 2008/2009

Problem Sheet 8

Discussion: January 14, 14:15 in Hörsaal 118, AVZ

1. Differential Forms

   Totally antisymmetric lower-index tensors are an important class of tensors, called differential forms. Given such a tensor $A_{\mu_1\dots\mu_p}$, antisymmetric in all its indices, the corresponding p-form A_p is defined as

   $$A_p = \frac{1}{p!}\, A_{\mu_1\dots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\mu_p}\,.$$

   Here the wedge product of the basis one-forms is antisymmetric, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. The wedge product extends to arbitrary forms,

   $$\begin{aligned} A_p \wedge B_q &= \frac{1}{p!}\frac{1}{q!}\, A_{\mu_1\dots\mu_p} B_{\nu_1\dots\nu_q} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\nu_p} \\ &= \frac{1}{(p+q)!}\left(A_p \wedge B_q\right)_{\mu_1\dots\mu_{p+q}} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\mu_{p+q}}\,. \end{aligned}$$

   Hence the components of the product form are given by (the square brackets indicate antisymmetrisation)

   $$\left(A_p \wedge B_q\right)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\, A_{[\mu_1\dots\mu_p} B_{\mu_{p+1}\dots\mu_{p+q}]}\,.$$

   Clearly, the degree of a form cannot exceed the spacetime dimension.

   One reason for the importance of forms is that they allow for a type of derivative which does not require a connection, the exterior derivative d. It increases the degree of the form and act as follows:

   $$\begin{aligned} \mathrm{d}A_p &= \mathrm{d}\left(\frac{1}{p!} A_{\mu_1\dots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\mu_p}\right) \\ &= \frac{1}{p!}\, \partial_\rho A_{\mu_1\dots\mu_p} \mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \mathrm{d}x^{\mu_p}\,. \end{aligned}$$

   In other words, the components of the resulting $(p+1)$-form are

   $$\left(\mathrm{d}A_p\right)_{\mu_1\dots\mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2\dots\mu_{p+1}]}\,.$$

   (a) Verify that the result of the exterior derivative is indeed a tensor! Furthermore, show that $\mathrm{d}^2 = 0$ and that the exterior derivative satisfies a Leibniz rule,

   $$\mathrm{d}\left(A_p \wedge B_q\right) = \mathrm{d}A_p \wedge B_q + (-1)^p\, A_p \wedge \mathrm{d}B_q\,.$$

(b) How many independent components does a p-form have in d spacetime dimensions? Given a (Lorentzian) metric, we can assign to a p-form A_p a $(d-p)$-form $(*A)_{d-p}$ with components

$$(*A)_{\mu_1\dots\mu_{d-p}} = \frac{1}{p!}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_d} g^{\mu_{d-p+1}\nu_1}\dots g^{\mu_d\nu_p} A_{\nu_1\dots\nu_p}$$

Here $\varepsilon_{\mu_1\dots\mu_d}$ is the totally antisymmetric Levi-Civita symbol, $\varepsilon_{012\dots d}=1$, and g is the determinant of the metric. Show that this is indeed a tensor! (It suffices to show that $\sqrt{-g}\varepsilon_{\mu_1\dots\mu_d}$ is a tensor, the so-called Levi-Civita tensor.) This operation is called Hodge-$*$. Compute the action of $**$!

(c) Specialise to three-dimensional Euclidean space. Consider a scalar function $\phi(x)$ and a vector field $\vec{u}(x)$ and express the usual operations grad, curl and div in form language. Derive the well-known identities

i. $\operatorname{curl}\operatorname{grad}\phi = 0$,
ii. $\operatorname{div}\operatorname{curl}\vec{u} = 0$,
iii. $\operatorname{curl}\operatorname{curl}\vec{u} = \operatorname{grad}\operatorname{div}\vec{u} - \Delta\vec{u}$.
iv. Let $\vec{v}$ be another vector field. Express the cross product $\vec{u}\times\vec{v}$ by forms.

2. (a) Show that the volume form V is $V = *1$. Show further that for two p-forms A_p and B_p, we have $A\wedge *B = B\wedge *A$.

(b) Consider Stokes' theorem

$$\int_V \mathrm{d}\omega = \int_{\partial V}\omega\,,$$

where ω is a d-form and V is a $d+1$-dimensional domain. What is the meaning of this theorem for $d=0,1,2$?

3. As is well known, electrodynamics is most naturally formulated in form language: The gauge field is a one-form A_1 with field strength $F_2 = \mathrm{d}A_1$, the gauge transformation being $A_1 \to A_1 + \mathrm{d}\Lambda_0$. The Lagrangean is given by $\mathscr{L} = F_2\wedge *F_2$. Recall (or convince yourself) that the resulting equation of motion is $\mathrm{d}*F_2 = 0$.

Maxwell's equations without sources are symmetric under exchange of electric and magnetic fields. How is this duality expressed in form language?

As an analogy, consider the theory of a free 2-form field B_2 with action

$$S = \int \mathrm{d}B\wedge *\mathrm{d}B$$

(a) What is the gauge invariance of this theory?

(b) Show that this theory is dual to a theory of a free scalar field with action (ignoring numerical prefactors)

$$S_{\text{dual}} = \int \mathrm{d}\phi\wedge *\mathrm{d}\phi\,.$$

Is there a remaining gauge symmetry?