Exercises on Theoretical Particle Physics

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H 5.1 The CKM matrix *3+1+0.5+1+1=6.5 points*

On the previous exercise sheet we have investigated the Standard Model (SM) Higgs effect on the gauge bosons. We have seen that the gauge bosons of the unbroken subgroup remain massless while those of the broken subgroup become massive. Since the SM fermions are chiral we can not write down a gauge invariant mass term. Therefore we have to employ the Higgs mechanism to generate fermion masses.

(a) Consider the following term of the Lagrangian in the one family approximation:

$$\mathscr{L} \supset \overline{R}\,(\mathrm{i}\gamma^\mu D_\mu)\,R + \overline{L}\,(\mathrm{i}\gamma^\mu D_\mu)\,L\,. \tag{1}$$

Find the interaction terms of the fermions with the gauge bosons. For the weak interactions, analyze its V-A structure $\frac{1}{2}(c_\mathrm{V} + c_\mathrm{A}\gamma^5)$. Draw the corresponding Feynman diagrams.
Hint: Use $\mathrm{i}\mathscr{L}$*. Drop all fields and you will get the vertex factor.*

(b) Show that the mass of the electron is $m_e = \frac{G_e\, v}{\sqrt{2}}$.
Hint: Using the unitary gauge, insert the shifted Higgs field $\Phi(x) = (0, \frac{1}{\sqrt{2}}(v+\eta(x)))^T$ *into the so-called Yukawa coupling of the Lagrangian* $\mathscr{L} \supset -G_e(\overline{L}\Phi R + \overline{R}\Phi^\dagger L)$.

(c) The situation can be generalized to the multi-family case. Consider the Yukawa couplings of N generations of quarks and leptons after spontaneous symmetry breaking

$$\mathscr{L} \supset -G_d^{(ij)}\overline{d}_{Li}d_{Rj} - G_u^{(ij)}\overline{u}_{Li}u_{Rj} + \mathrm{h.c.}\,, \tag{2}$$

where the indices i and j denote the generation and we sum over repeated indices. Why do we have to include the " + h.c." part?

The real matrices G_u and G_d do not need to be diagonal. So, the quarks u and d are not mass eigenstates, but are eigenstates of the weak interaction. Nevertheless, only mass eigenstates are regarded as physical particles that can be detected in an experiment. Thus one has to perform a basis transformation and diagonalize the mass matrices.

(d) Use biunitary transformations $S_d\, G_d\, T_d^\dagger = G_d^\mathrm{diag}$ and $S_u\, G_u\, T_u^\dagger = G_u^\mathrm{diag}$, where $S_{u/d}$ and $T_{u/d}$ are unitary matrices, to diagonalize the mass matrices. What happens to the quarks during the diagonalization?

(e) We would like to analyze how this change of basis affects the weak interaction. The relevant term in the Lagrangian is

$$\mathscr{L} \supset \overline{L}_i \left(\mathrm{i}\gamma^\mu D_\mu\right) L_i \tag{3}$$
$$\supset -\frac{g}{\sqrt{2}} \left(W_\mu^+ \, \overline{u}_{Li} \, \gamma^\mu \, d_{Li} + W_\mu^- \, \overline{d}_{Li} \, \gamma^\mu \, u_{Li}\right), \tag{4}$$

with $L_i = (u_{Li}, d_{Li})^T$. Use the mass eigenstates of quarks to investigate the weak interaction vertex.

H 5.2 Majorana spinors and the see–saw *0.5+1+1+1+1.5+1+1+3.5=10.5 points*

We write a four component Dirac spinor in the chiral representation as a composition of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix} . \tag{5}$$

A *Majorana spinor* is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\overline{\Psi}^T = \Psi \,, \tag{6}$$

where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation operator.

(a) Show that $(\Psi^c)^c = \Psi$.

(b) What does eq. (6) imply for ψ_L and ψ_R and what is the physical meaning of this condition?

(c) The Lagrangian $\mathscr{L}_\mathrm{D}$ for a Dirac spinor has the form

$$\mathscr{L}_\mathrm{D} = \overline{\Psi} \left(\mathrm{i}\gamma^\mu \partial_\mu\right) \Psi - m\overline{\Psi}\Psi \,, \tag{7}$$

where the second term is called the *Dirac mass term.* Rewrite $\mathscr{L}_\mathrm{D}$ in ψ_L and ψ_R

(d) Using the result of (b) rewrite the action $\mathscr{L}_\mathrm{M}$ for a Majorana spinor in terms of $\psi_\mathrm{L/R}$

$$\mathscr{L}_\mathrm{M} = \overline{\Psi} \left(\mathrm{i}\gamma^\mu \partial_\mu\right) \Psi - \frac{m}{2}\overline{\Psi}\Psi. \tag{8}$$

The second term is called the *Majorana mass term.* Why is the factor 1/2 included in the mass term?

(e) Remember the projectors $P_\mathrm{L/R} = 1/2(\mathbb{1} \mp \gamma^5)$. As you know $P_\mathrm{L/R}$ project Ψ to the left/right handed part, respectively. We denote $\Psi_\mathrm{L/R} := P_\mathrm{L/R}\Psi$. Show that

$$\left(\Psi_\mathrm{L/R}\right)^c = \left(\Psi^c\right)_\mathrm{R/L} \tag{9}$$
$$\overline{\left(\Psi_\mathrm{L/R}\right)^c} \left(\Psi_\mathrm{R/L}\right)^c = \overline{\Psi_\mathrm{R/L}}\Psi_\mathrm{L/R} \tag{10}$$

(f) The most general mass term for a Dirac spinor is the *Dirac–Majorana mass term*,

$$\mathscr{L}_{\mathrm{m}} = -\frac{1}{2}\left[2m_{\mathrm{D}}\overline{\Psi_{\mathrm{L}}}\Psi_{\mathrm{R}} + m_{\mathrm{L}}\overline{\Psi_{\mathrm{L}}}(\Psi^c)_{\mathrm{R}} + m_{\mathrm{R}}\overline{(\Psi^c)_{\mathrm{L}}}\Psi_{\mathrm{R}}\right] \tag{11}$$

Show that this can be written in matrix form as

$$\mathscr{L}_{\mathrm{m}} = -\frac{1}{2}\begin{pmatrix}\overline{\Psi_{\mathrm{L}}} & \overline{(\Psi^c)_{\mathrm{L}}}\end{pmatrix}\mathcal{M}\begin{pmatrix}(\Psi^c)_{\mathrm{R}} \\ \Psi_{\mathrm{R}}\end{pmatrix} \tag{12}$$

with

$$\mathcal{M} = \begin{pmatrix} m_{\mathrm{L}} & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_{\mathrm{R}}\end{pmatrix} \tag{13}$$

being the *neutrino mass matrix*.

(g) Argue that in the SM extended by right-handed neutrinos, m_{L} must be zero and m_{D} is of the order of the electroweak symmetry breaking scale $M_W \sim 100\,\mathrm{GeV}$. We further assume that m_{R} is generated by some unspecified symmetry breaking mechanisms occurring at high energies, i. e. $m_{\mathrm{R}} \sim M_{\mathrm{GUT}} \sim 10^{16}\,\mathrm{GeV}$.

(h) In this setup, diagonalize $\mathcal{M}$ using an orthogonal matrix A

$$A^T\mathcal{M}A = \mathrm{diag}(m_1, m_2). \tag{14}$$

Show that to the first non-vanishing order in the (small) parameter $\rho := m_{\mathrm{D}}/m_{\mathrm{R}}$ that the eigenvalues are $m_1 = -m_{\mathrm{D}}^2/m_{\mathrm{R}}$ and $m_2 = m_{\mathrm{R}}$. Find the rotation matrix A to the first order in ρ for the diagonalization. What does $\rho \ll 1$ imply for the mass eigenstates? Insert the estimations done in (g) and compare the mass of the light neutrino to actual experimental bounds.

We see that by making one mass heavy the other one becomes very light. For this reason setups of this kind are generically referred to as *See-Saw mechanism*.