Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 6
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Exercises on Theoretical Particle Physics

Prof. Dr. H.-P. Nilles

H 6.1 Neutrino oscillations

0.5+1+1.5+1+2=6 points

In analogy to the mixing of the quarks through weak interactions via the CKM matrix one can imagine a similar situation with the leptons once the neutrinos get mass. Hence, let us assume that there are n orthonormal flavor (interaction) eigenstates $|\nu_\alpha\rangle$. These states are transformed into n mass eigenstates ν_i via the unitary mixing matrix U,

$$|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle\,. \tag{1}$$

(a) Assuming that the mass eigenstates $|\nu_\mathrm{i}\rangle$ are stationary states and were emitted with momentum p by a source at $x = 0$ at $t = 0$, what is the form of $|\nu_i(x,t)\rangle$?

(b) What is the relativistic Hamiltonian for a particle? For a highly relativistic particle we have $m \ll p$. Expand the Hamiltonian to first non-vanishing order in m/p.

(c) A neutrino detector is built at a distance L from a source producing neutrinos in an eigenstate $|\nu_\alpha\rangle$. Show that the amplitude of detecting a neutrino in an eigenstate $|\nu_\beta\rangle$ is

$$A(\alpha\to\beta)(L) = \sum_i U^*_{\beta i}U_{\alpha i}\,\exp\left\{\mathrm{i}\,\frac{m_i^2}{2}\,\frac{L}{E}\right\}\,. \tag{2}$$

Hint: For a highly relativistic particle you can set $v = 1(= c)$ *and* $p \cong E$.

(d) Obtain the transition probability P in terms of the mass square differences $\Delta m^2_{ij} := m_i^2 - m_j^2$. What is the probability of finding the original flavor?

(e) Now assume that we have two flavors and one mixing angle θ. What is the form of U? Compute $P(\alpha\to\beta)$ and $P(\alpha\to\alpha)$ for this case. Under which condition can one flavor completely rotate into another one?

H 6.2 Dynkin diagram of $\mathfrak{su}(N)$ *1.5+1+0.5+1.5+1.5+1+1.5+0.5+2=11 points*

Consider the space of all $N \times N$ matrices and regard it as a Lie algebra $\mathfrak{gl}(N)$. We choose as a basis the elements e_{ab} with components $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

(a) Verify the multiplication rule and thus the commutator operation on the algebra

$$e_{ab}e_{cd} = e_{ad}\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad}. \tag{3}$$

In order to deal with the Lie algebra $\mathfrak{su}(N)$, what restrictions have to be made?. Write down a basis for $\mathfrak{su}(N)$. What is the dimension?

(b) The **Cartan algebra** $\mathfrak{h}$ is defined to be a maximal commuting subalgebra of the Lie algebra. Its dimension is called the **rank** of the Lie algebra. Give a possible choice for the Cartan subalgebra of $\mathfrak{su}(N)$. What is the rank r of $\mathfrak{su}(N)$?

(c) Now we want to diagonalize the Cartan algebra in the adjoint representation which acts by the commutator

$$\operatorname{ad} h(g) = [h, g] \tag{4}$$

Perform a (complex) basis change of $\mathfrak{su}(N)/\mathfrak{h}$ to an eigenbasis of $\mathfrak{h}$. You should find,

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab}\,, \tag{5}$$

with $h = \sum_i \lambda_i e_{ii}$.

We can regard eq. (5) (for e_{ab} fixed) as a prescription for how to associate a number $(\lambda_a - \lambda_b)$ to each $h \in \mathfrak{h}$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b. \tag{6}$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra $\mathfrak{h}$. This dual space is denoted by $\mathfrak{h}^*$.

Let $\alpha_1 \ldots \alpha_r$ be a fixed basis of roots so every element of $\mathfrak{h}^*$ can be written as $\rho = \sum_i c_i\alpha_i$.

We call ρ **positive** ($\rho > 0$) if the first non-zero coefficient c_i is positive. Note, that the basis roots α_i are positive by definition. If the first non-zero coefficient c_i is negative, we call ρ negative. For $\rho, \sigma \in \mathfrak{h}^*$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$. A **simple root** is a positive root which can not be written as the sum of two positive roots.

(d) We choose a basis α_i for the root space:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1}, \quad i = 1, 2, \ldots, N-1\,. \tag{7}$$

Verify that these roots are a basis and that they are positive with $\alpha_1 > \alpha_2 > \ldots > \alpha_{N-1}$. Show that these roots are simple roots.

Next, we define a structure that resembles a scalar product on the algebra. Let t_i be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra:

$$[x,[y,t_i]] = \sum_j K_{ij} t_j \,. \tag{8}$$

The **Killing form** is then defined as $\mathcal{K}(x,y) := \mathrm{Tr}(K)$.

(e) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h,h')$, where $h = \sum_i \lambda_i e_{ii}$, $h' = \sum_j \lambda'_j e_{jj}$.

The Killing form enables us to make a connection between the Cartan subalgebra, $\mathfrak{h}$, and its dual $\mathfrak{h}^*$: One can prove that if $\alpha \in \mathfrak{h}^*$, there exists a unique element $h_\alpha \in \mathfrak{h}$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \quad \forall h \in \mathfrak{h} \,. \tag{9}$$

(f) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find h_{α_i} from comparison with your result from (e).

With the help of the h_α, we are now able to define a scalar product on $\mathfrak{h}^*$:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \quad \text{where } \alpha_i, \alpha_j \in \mathfrak{h}^*. \tag{10}$$

(g) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \,. \tag{11}$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root α_i, we associate a small circle and join the small circles i and j with $A_{ij}A_{ij}$ (no summation, $i \neq j$) lines.

(h) Draw the Dynkin diagram for $\mathfrak{su}(N)$.

(i) As an example, consider the Lie algebra of $\mathfrak{su}(2)$. The step operators are given by

$$J_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2) \,, \tag{12}$$

and the Cartan subalgebra consists of the single element

$$h = J_3 = \frac{1}{2}\sigma_3 \,. \tag{13}$$

(i.1) Confirm that

$$e_{12} = J_+, \;\; e_{21} = J_- \;\; \text{and} \;\; h = \frac{1}{2}e_{11} - \frac{1}{2}e_{22} \,. \tag{14}$$

(i.2) Calculate $\alpha_{J_\pm}(J_3)$.

(i.3) Choose $\alpha_1 = \alpha_{J_+}$ as the basis root, which is positive and simple. For $\alpha_1 \in \mathfrak{h}^*$, find the unique element $h_{\alpha_1} \in \mathfrak{h}$ such that

$$\alpha_1(h) = \mathcal{K}(h_{\alpha_1}, h) \quad \forall h \in \mathfrak{h} \,. \tag{15}$$

Hint: The solution is $h_{\alpha_1} = \frac{1}{2} J_3$.

(i.4) Calculate the Killing form $\mathcal{K}(h_{\alpha_1}, h_{\alpha_2})$ and draw the Dynkin diagram.