Physikalisches Institut
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Theoretische Physik

Exercise 1
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WS 10/11

Exercises on Theoretical Particle Physics

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–Home Exercises–
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H 1.1 The Lorentz group

1+2+3+3+1=10 points

The Lorentz group is defined as the set of transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

which leave the scalar product $\langle x, y\rangle = \eta_{\mu\nu}\, x^\mu y^\nu$ invariant.

(a) Show that an element λ of the Lie algebra of the Lorentz group satisfies:

$$\lambda^T = -\eta\lambda\eta\,.$$

Hint: Reformulate the statement about the invariance of the scalar product in $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu$ *and write an element of the Lorentz group as* $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - i\lambda^\mu{}_\nu$.

(b) Choose

$$(M^{\mu\nu})^\rho{}_\sigma = \mathrm{i}\,(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)$$

as a basis for the Lie algebra. What do these matrices look like? Describe the form of the matrices in words. Verify the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\,(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho})\ .$$

(c) We split the generators into two groups:

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}\,, \qquad K^i = M^{0i}\,.$$

The J's have only spatial indices, the K's have spatial and timelike indices. Verify the commutation relations

$$\left[J^i, J^j\right] = \mathrm{i}\,\epsilon^{ijk}J^k\,, \qquad \left[J^i, K^j\right] = \mathrm{i}\,\epsilon^{ijk}K^k\,, \qquad \left[K^i, K^j\right] = -\mathrm{i}\,\epsilon^{ijk}J^k\,,$$

and describe the meaning of each relation in words. What kind of transformations do the J's and K's correspond to?

(d) The form of the commutation relations for the Lorentz algebra can still be simplified. Define

$$T^i_{\mathrm{L/R}} = \frac{1}{2}\left(J^i \pm \mathrm{i}\, K^i\right)$$

and verify the commutation relations

$$\left[T^i_{\mathrm{L}}, T^j_{\mathrm{L}}\right] = \mathrm{i}\,\epsilon^{ijk}\, T^k_{\mathrm{L}}\,, \qquad \left[T^i_{\mathrm{R}}, T^j_{\mathrm{R}}\right] = \mathrm{i}\,\epsilon^{ijk}\, T^k_{\mathrm{R}}\,, \qquad \left[T^i_{\mathrm{L}}, T^j_{\mathrm{R}}\right] = 0\,.$$

(e) Classify the representations of the Lorentz algebra using what you learned about $\mathfrak{su}(2)$.

Conclusion: Every representation of the Lorentz algebra can be characterized by two non-negative integers or half-integers (j_{L}, j_{R}).

H 1.2 γ-Matrix identities

1.5+5+3.5=10 points

The following exercise is to be solved by only using the Clifford algebra of the γ-matrices and **not** a particular representation. For convenience we introduce the notation

$$\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,.$$

(a) Show that

$$\left(\gamma^5\right)^\dagger = \gamma^5\,, \qquad \left(\gamma^5\right)^2 = \mathbb{1}\,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0\,.$$

(b) Prove the following trace theorems.

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4\eta^{\mu\nu} \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \\
\operatorname{tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\right) &= 0\,, \qquad \text{for } n \text{ odd} \\
\operatorname{tr}\gamma^5 &= 0 \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \\
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) &= -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}
\end{aligned}$$

Hint: Use the cyclicity of the trace.

(c) Show the following contraction identities:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4\cdot\mathbb{1} \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho}\,\mathbb{1} \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu
\end{aligned}$$