Physikalisches Institut
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Theoretische Physik

Exercise 10
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WS 10/11

Exercises on Theoretical Particle Physics

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–Home Exercises–

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H 10.1 Renormalization of the Electric Charge in QED

19 points

We calculate loop corrections to the photon propagator in QED due to the vacuum polarization diagram. We will see that the correction can be interpreted as a renormalization effect on the electric charge, the QED coupling constant. The vacuum polarization diagram is given by the (amputated) Feynman diagram given in fig. 1

(a) Write down the matrix element $\mathrm{i}\Pi^{\mu\nu}$ for this process. Use the QED Feynman rules from Ex. 4.2 plus the additional Feynman rules tab. 1 . You will find

$$\mathrm{i}\Pi^{\mu\nu}(q) = -e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \,\mathrm{tr}\left(\gamma^\mu \frac{\not k + m}{k^2 - m^2 + \mathrm{i}\epsilon} \gamma^\nu \frac{\not k + \not q + m}{(k+q)^2 - m^2 + \mathrm{i}\epsilon}\right) . \qquad (1)$$

Hint: The trace comes from the contraction of the spinor indices of the γ-matrices. (*1 point*)

(b) Use the trace theorems for γ-matrices to simplify the numerator of eqn. (1). (*1 point*)

(c) Prove the so-called Feynman trick:

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \frac{1}{[xa + (1-x)b]^2} .$$

(*1 point*)

(d) Use the Feynman trick to combine the two denominators of eqn. (1). The result reads

$$\int_0^1 \mathrm{d}x \frac{1}{[l^2 + x(1-x)q^2 - m^2 + \mathrm{i}\epsilon]^2} ,$$

where $l = k + xq$. (*1 point*)

(e) Shift the integration variable from an integration over k to an integration over l and argue that you can drop all terms linear in l. The result is:

$$\mathrm{i}\Pi^{\mu\nu}(q) = -4e^2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{2l^\mu l^\nu + 2x(x-1)q^\mu q^\nu - g^{\mu\nu} l^2 - g^{\mu\nu}(x(x-1)q^2 - m^2)}{(l^2 - \Delta + \mathrm{i}\epsilon)^2} , \qquad (2)$$

where $\Delta = m^2 - x(1-x)q^2$. (*1.5 points*)

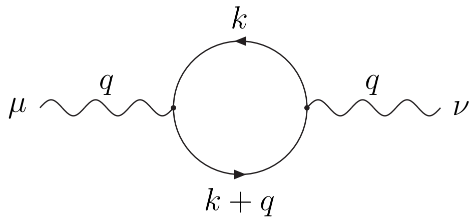

Figure 1: Vacuum Polarization Feynman Graph

(f) In QED one can prove that, due to the gauge symmetry, all terms proportional to q^μ or q^ν vanish in every S-matrix calculation. Drop the corresponding term from your result. (The proof makes use of the so-called *Ward Identity* of QED.) (*0.5 points*)

(g) Show that

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{l^\mu l^\nu}{f(l^2)} = \frac{1}{4} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} g^{\mu\nu} \frac{l^2}{f(l^2)} .$$

(*1 point*)

(h) Recall that $l^2 = (l^0)^2 - (l^i)^2$. Therefore, the integral of eqn. (2) is one over a Minkowski space. It is much more convenient to perform such integrals in 4-dim Euclidean space. To do so, one has to perform a *Wick rotation*:

(i) View l^0 as a complex variable. Draw the complex l^0-plane. The integration is along the real axis. Mark the position of the poles of eqn. (2).

(ii) Use Cauchy's integral theorem to argue that the integral from $-\infty$ to $+\infty$ is equal to the integral from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

(iii) So define new (Euclidean) coordinates: $l^0 = \mathrm{i}n^0$ and $l^i = n^i$ and rewrite the integral n terms of n^μ. At the end, rename n^μ to l^μ.

(iv) Now we can set $\epsilon \to 0$, because there is no divergence on the path of integration.

The result should read:

$$\mathrm{i}\Pi^{\mu\nu}(q) = -4\mathrm{i}e^2 g^{\mu\nu} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{\frac{1}{2}l^2 + x(1-x)q^2 + m^2}{(l^2 - \Delta)^2} , \qquad (3)$$

(*2 points*)

Now we will solve the integral and interpret the resulting correction of the photon propagator as a renormalization of the electric charge.

(i) Prove that $\int \mathrm{d}\Omega_4 = 2\pi^2$. *Hint: Multiply the known integrals* $\int_{-\infty}^{\infty} dl_i e^{-l_i^2} = \sqrt{\pi}$ *for* $i = 0, ..., 3$ *and change from Cartesian coordinates to 4-dim. spherical coordinates* $d^4 l = |l|^3 d|l| d\Omega_4$. *Then substitute* $z = |l|^2$ *and solve the remaining integral using partial integration.* (*1 point*)

Feynman Propagator of Fermions with Momentum q	$\mathrm{i}\frac{\not{q}+m}{q^2-m^2+\mathrm{i}\epsilon}$
Loop momentum k	$\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$
Fermion loop	$\cdot(-1)$

Table 1: QED Feynman rules II

(j) In Euclidean space we can now change eqn. (3) to polar coordinates. Perform the substitution $z = |l|^2$. *(1 point)*

(k) Next, we want to solve the integrals over z. Therefore, perform the following integrations:

$$\int_a^b \frac{z^2\,\mathrm{d}z}{(z+\Delta)^2} = \left(z - 2\Delta\log z - \frac{\Delta^2}{z}\right)_{a+\Delta}^{b+\Delta}, \qquad \int_a^b \frac{z\,\mathrm{d}z}{(z+\Delta)^2} = \left(\log z + \frac{\Delta}{z}\right)_{a+\Delta}^{b+\Delta}.$$

Using the boundaries from 0 to $+\infty$, we see that they are divergent. We regularize them by an energy cutoff, i.e. we integrate from 0 to Λ^2. Note: $z = |l|^2 = |k + xq|^2$, so the momentum k in the loop only runs up to an upper limit. *(1 point)*

(l) Verify that in the limit of large Λ the following approximations hold

$$\int_0^{\Lambda^2} \frac{z^2}{(z+\Delta)^2}\mathrm{d}z \to \Lambda^2 - 2\Delta\log\frac{\Lambda^2}{\Delta} + \Delta\,, \qquad \int_0^{\Lambda^2} \frac{z}{(z+\Delta)^2}\mathrm{d}z \to \log\frac{\Lambda^2}{\Delta} - 1$$

in order to obtain

$$\mathrm{i}\Pi^{\mu\nu}(q) = -\frac{\mathrm{i}e^2}{4\pi^2}g^{\mu\nu}\int_0^1 \mathrm{d}x\left\{\frac{1}{2}\left(\Lambda^2 - 2\Delta\log\frac{\Lambda^2}{\Delta} + \Delta\right) + [x(1-x)q^2 + m^2]\left(\log\frac{\Lambda^2}{\Delta} - 1\right)\right\}.$$

(1.5 points)

(m) This result is not gauge invariant, because the cutoff regularization does not respect the QED symmetry. Restore the symmetry by discarding all terms that are not proportional to q^2. (The terms not proportional to q^2 would give rise to a photon mass which is not allowed by the gauge symmetry.) *(1 point)*

(n) Choose the cutoff to be extremely large (of the order of the GUT scale), so we can assume that the cutoff is much larger than the external momentum q, i.e. $\Lambda^2 \gg q^2$. *(0.5 points)*

(o) Next, we consider two limits: (i) q^2 small and (ii) q^2 large.

   (i) q^2 small – In this limit, we define the measurable value of the electric charge. Use $m^2 \gg x(1-x)q^2$ to prove the final result for the matrix element:

$$\mathrm{i}\Pi^{\mu\nu}(q) = \frac{\mathrm{i}e^2}{12\pi^2} g^{\mu\nu} q^2 \log\frac{m^2}{\Lambda^2}.$$

We can now use this result to calculate the loop corrected photon propagator. Calculate the correction at one loop and follow that the propagator is given by

$$-\frac{\mathrm{i}g^{\mu\nu}}{q^2}\left[1+\frac{e^2}{12\pi^2}\log\frac{m^2}{\Lambda^2}\right].$$

Now calculate the correction to all orders (several one-loop diagrams one after another). Using the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

you will obtain

$$-\frac{\mathrm{i}g^{\mu\nu}}{q^2}\left[\frac{1}{1-\frac{e^2}{12\pi^2}\log\frac{m^2}{\Lambda^2}}\right] =: -\frac{\mathrm{i}g^{\mu\nu}}{q^2} Z_3.$$

As every propagator ends in two vertices, we can also use our original propagator and multiply $\sqrt{Z_3}$ to each vertex $ie\gamma^\mu$ instead. Thus, we can regard $\sqrt{Z_3}$ as a factor multiplying the electromagnetic charge which gives the *renormalized charge* or *renormalized coupling constant*: $e_R := \sqrt{Z_3} e$. Note that it is the renormalized charge that is measured in experiments. In order to distinguish the renormalized (physical) charge from the original parameter e in the Lagrangian, we speak of e as the *bare charge* or *bare coupling constant*.

(ii) q large – In this limit, we can calculate the dependence of the charge e on the momentum q. First, write the logarithm as:

$$\log\left(\frac{\Lambda^2}{m^2 - x(1-x)q^2}\right) = -\log\left(-\frac{q^2}{\Lambda^2}\right) - \log(x(1-x)) - \log\left(1 - \frac{m^2}{q^2 x(1-x)}\right)$$

The last term vanishes for $q^2 \gg m^2$. For the x-integration, you need:

$$\int_0^1 dx\; x(1-x)\log(x(1-x)) = -\frac{5}{18}$$

Show that the final result for the matrix element reads:

$$\mathrm{i}\Pi^{\mu\nu}(q) = \frac{\mathrm{i}e^2}{12\pi^2} g^{\mu\nu} q^2 \left(\log\left(-\frac{q^2}{\Lambda^2}\right) - \frac{5}{3}\right)$$

Following the discussion of part (1) you find:

$$e_R(q) = \frac{e}{1 - \frac{e^2}{12\pi^2}\left(\log\left(-\frac{q^2}{\Lambda^2}\right) - \frac{5}{3}\right)}$$

(4 points)