Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 01
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WS 11/12

Exercises on String Theory I

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–Home Exercises–
Due 07 November 2011

Exercise 2.1: Virasoro Algebra (*16 credits*)

In this exercise we investigate the Virasoro algebra. Recall that you derived on the last sheet the following algebra of the α's:

$$[\alpha_m^\mu, \alpha_n^\nu]_{\text{PB}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\text{PB}} = im\eta^{\mu\nu}\delta_{m+n,0}\,, \qquad [\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{\text{PB}} = 0\,.$$

(a) Use the above algebra to derive the Virasoro algebra

$$[L_m, L_n]_{\text{PB}} = \mathrm{i}\,(m-n)\,L_{m+n}\,,$$

where the generators L_m are defined as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n\,.$$

Note that at this point the Poisson brackets are still classical, so you need not worry about the ordering of the operators. (*3 credits*)

Next, we deal with the quantized Virasoro algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + A(m)\,\delta_{m+n,0}\,,$$

which includes the central charge (or anomaly) term $A(m)$.

(b) Why does this term only arise for $m+n=0$? (*2 credits*)

(c) Use the Jacobi identity for the L_m to derive the recursion relation (*2 credits*)

$$A(m+1) = \frac{m+2}{m-1}A(m) - \frac{2m+1}{m-1}A(1)\,.$$

(d) Argue that this relation implies that $A(m) = cm + dm^3$! Here, c and d are some real coefficients. You may assume that $A(m)$ is a polynomial. (*3 credits*)
Hint: Exploit the linearity of the recursion relation!

(e) Determine c and d by evaluating the expectation value

$$\langle 0 | [L_m, L_{-m}] | 0 \rangle$$

in a zero momentum ground state $|0\rangle$ for some suitable values of m. (*6 credits*)

Exercise 2.2: Dimensionality of the bosonic string (*4 credits*)

In this exercise we show that $D = 26$ is part of the boundary of the ghost–free subspace of the Hilbert space using spurious states. Physical spurious (and hence null) states appear for $a = 1$, as shown in the lecture by considering spurious states of the form $|\psi\rangle = L_{-1} |\chi_1\rangle$. Now consider a different state defined as

$$|\psi\rangle = \left(L_{-2} + \gamma L_{-1}^2\right) |\chi\rangle \ ,$$

where γ is some number.

(a) Derive the conditions the state $|\chi\rangle$ has to satisfy such that $|\psi\rangle$ is spurious. (*2 credits*)

(b) Show that by requiring $|\psi\rangle$ to be physical, i.e. by imposing the Virasoro constraints, we obtain $\gamma = \frac{3}{2}$ and $D = 26$. (*2 credits*)