Exercises on String Theory I

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–Home Exercises–
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This exercise sheet is devoted to the study of anomalies. Anomalies are classical symmetries of a theory which are broken by quantum effects. The presence of (non–global) anomalies renders a theory inconsistent. For this reason, anomalies have to be absent. Unfortunately, ensuring the absence of anomalies is highly non–trivial. If it was not for String Theory, which is anomaly free by construction, it would be extremely hard (or even impossible) to construct consistent higher dimensional anomaly free theories.

Exercise 4.1: Anomalies in 4 dimensions (*10 credits*)

Consider a 4 dimensional theory with (Abelian or non–Abelian) gauge fields A^a_μ and N left–chiral Weyl fermions Ψ_i with gauge charges q^a_i. In 4 dimensions the Feynman graph responsible for the anomaly is given in figure 1. Here, $j^a_\mu = \frac{\delta S}{\delta A^{a,\mu}}$ is the current coupling to the gauge field A^a_μ and a, b, c label various gauge symmetries which can occur in the theory. The T^a stand either for the Abelian charges q^a or the non–Abelian generators in the respective representation. The particles running in the loop are all N chiral fermions in the theory. The graph leads to an anomalous variation of the path integral measure which leads to an effective change of the action like

$$\delta S_{\text{anom}} \propto \int \mathrm{d}^4 x \lambda^a F^b \wedge F^c\,,$$

where λ^a is the gauge parameter.

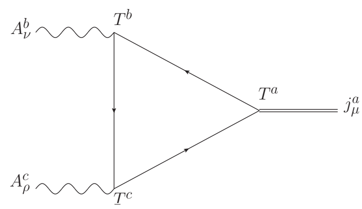


Figure 1: 4D triangle anomaly graph

(a) We first discuss the case where all gauge symmetries are Abelian. Show that including the charges q_i^a of the fermions in the graph, the cancelation of the anomaly leads to the condition

$$\sum_{i=1}^{N} q_i^a q_i^b q_i^c = 0\,. \tag{1}$$

Why is it sufficient to consider only massless fermions? (*2 credits*)

(b) Take the familiar Standard Model of particle physics. Show that (1) is indeed fulfilled for $T^a = T^b = T^c = T_Y$ where T_Y is the hypercharge generator. (*2 credits*)

(c) Next, consider one $U(1)$ and one non–Abelian symmetry (e.g. SU(N)) with the particles transforming only in the trivial or in the fundamental and anti–fundamental representation. Show that including group theory factors in the Feynman graphs leads to the constraint

$$\sum_{i=1}^{N} l(\mathbf{R})\, q_i^a = 0\,.$$

Here $l(\mathbf{R})$ is the quadratic Casimir in the respective representation, i.e.

$$l(\mathbf{R}) = \mathrm{tr}_{\mathbf{R}} T^a T^a$$

Why is there no constraint containing two Abelian charges? (*2 credits*)

(d) Check by inserting the proper quantum numbers that the $U(1)_{\mathrm{Y}} - SU(2)_{\mathrm{L}} - SU(2)_{\mathrm{L}}$ anomaly vanishes in the Standard Model. (*2 credits*)

(e) Finally we replace two gauge fields A_ρ^d by universal graviton couplings. Show that this leads to the constraint

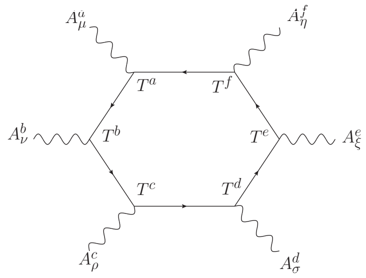
$$\sum_{i=1}^{N} q_i^a = 0\,.$$

Check that this is also fulfilled in the Standard Model. (*2 credits*)

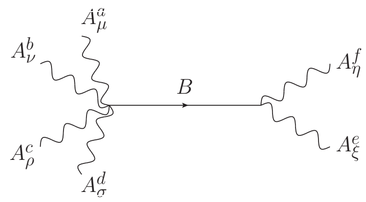
Exercise 4.2: Anomalies in 10 dimensions for Type I Sting Theory (*10 credits*)

In this exercise, we motivate that the only chance you have for canceling the 10 dimensional anomaly arising from the hexagonal graph with one simple Lie group as gauge group is $SO(32)$. The relevant anomaly graph in 10 dimensions is the hexagonal graph (which is analogous to the triangle graph in 4 dimensions). Again, the external legs are gauge bosons, and all massless particles run in the loop. The graph is given in figure 2a. By itself, the hexagon anomaly is always present. It is only because of the contribution of another diagram (the so–called Green–Schwarz counter term) given in figure 2b that there is the possibility of canceling the anomaly. The counter term arises from an axionic coupling of the Kalb–Ramond B–field. The anomaly given in figure 2a contains a factor $\mathrm{tr}_{\mathbf{Ad}} F^6$ where F is the field strength and $\mathrm{tr}_{\mathbf{Ad}}$ is the trace in the adjoint representation in which the gauge fields transform. However, the counter term in figure 2b does not contain $\mathrm{tr}_{\mathbf{Ad}} F^6$

contributions but only $\mathrm{tr}_{\mathbf{Ad}}F^4\ \mathrm{tr}_{\mathbf{Ad}}F^2$ contributions. Hence, if we want to have a chance of canceling the anomaly, $\mathrm{tr}_{\mathbf{Ad}}F^6$ has to factorize such that the pure F^6 term is absent. Checking this factorization is best done by relating the trace $\mathrm{tr}_{\mathbf{Ad}}$ in the adjoint to the trace $\mathrm{tr}_{\mathbf{2N}}$ in the fundamental (vector) representation. We will do so for $SO(2N)$ gauge groups.



(a) Hexagon anomaly graph



(b) Green–Schwarz counter term

(a) The traces in the different representations are related most easily using Chern characters. The Chern character is defined as

$$\mathrm{Ch}_{\mathbf{R}}(F) := \mathrm{tr}_{\mathbf{R}} \exp(F)\,.$$

Here, $\mathbf{R}$ stands for any representation of the group, and $\mathrm{tr}_{\mathbf{R}}$ is the trace in representation $\mathbf{R}$. Show that *(2 credits)*

$$(i) \qquad \mathrm{Ch}_{\mathbf{R_1}\otimes\mathbf{R_2}}(F) = \mathrm{Ch}_{\mathbf{R_1}}(F) \cdot \mathrm{Ch}_{\mathbf{R_2}}(F)\,,$$
$$(ii) \qquad \mathrm{Ch}_{\mathbf{R_1}\oplus\mathbf{R_2}}(F) = \mathrm{Ch}_{\mathbf{R_1}}(F) + \mathrm{Ch}_{\mathbf{R_2}}(F)\,.$$

(b) For $SO(2N)$ groups the adjoint representation $\mathbf{Ad}$ is given by the anti–symmetrized product of two fundamental representations $\mathbf{2N}$. Anti–symmetrization can be ensured by taking the determinant. The k–fold anti–symmetrized representation $[\mathbf{R}]_k$ can thus be obtained from the generating function

$$\sum_{k=1}^{N} x^k\, \mathrm{Ch}_{[\mathbf{R}]_k}(F) = \det{}_{\mathbf{R}}(1 + x e^F) = \exp\left[\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n\, \mathrm{Ch}_{\mathbf{R}}(nF)\right]\,.$$

By expanding the relation to the relevant order, show that *(3 credits)*

$$\mathrm{Ch}_{[\mathbf{2N}]_2}(F) = \mathrm{Ch}_{\mathbf{Ad}}(F) = \frac{1}{2}\left[(\mathrm{Ch}_{\mathbf{2N}}(F))^2 - \mathrm{Ch}_{\mathbf{2N}}(2F)\right]\,. \tag{2}$$

(c) By expanding (2) to the relevant powers, show that *(5 credits)*

$$\mathrm{tr}_{\mathbf{Ad}}F^6 = (2N-32)\mathrm{tr}_{\mathbf{2N}}F^6 + 15\,\mathrm{tr}_{\mathbf{2N}}F^4\ \mathrm{tr}_{\mathbf{2N}}F^2\,.$$

Hence the first term is only absent for $SO(32)$. As the second term is always non–zero, the hexagon anomaly never vanishes, so the Green–Schwarz counter term is really essential for obtaining a consistent theory. It is of course still highly non–trivial to check that all coefficients match such that the counter term does indeed cancel the $\mathrm{tr}_{\mathbf{2N}}F^4\ \mathrm{tr}_{\mathbf{2N}}F^2$ term, and that the mixed and pure gravitational anomalies are absent for $SO(32)$ as well. Similarly, one can show that the first term never vanishes for $USp(N)$ and $U(N)$ groups.