Exercises on String Theory I

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–Home Exercises–
Due 29. November 2011

Exercise 5.1: String on a Circle *7 Credits*

We consider a bosonic string coordinate compactified on a circle. For this we find new boundary conditions of the form

$$X^9(\tau, \sigma + \pi) = X^9(\tau, \sigma) + 2\pi R w\,, \tag{1}$$

where $w \in \mathbb{Z}$ is refered to as the **winding number**.

1. Show that for single valuedness of $e^{\mathrm{i}Xp}$, the momentum p^9 must be quantized as $p^9 = k/R$ with $k \in \mathbb{Z}$. (*1 credit*)
2. Consider the mode expansion of X^9,
$$X^9_{L/R}(\sigma_\pm) = x^9_{L/R} + p^9_{L/R}\sigma_\pm + \text{ oscillators }. \tag{2}$$
Express the left- and right moving momenta in terms of k and w. (*1 credit*)
3. Show that the mass condition $M^2 = M_L^2 + M_R^2$ and the level matching condition $M_L^2 = M_R^2$, with $M_{L/R}^2 = 1/2p_{L/R}^2 + N_{L/R} - 1$ become
$$M^2 = \frac{m^2}{4R^2} + w^2R^2 + N_L + N_R - 2\,, \tag{3}$$
$$0 = N_L - N_R + kw\,. \tag{4}$$
(*2 credits*)
4. Determine the massless spectrum. Show that you get additional massless states for $R^2 = 1/2$. (*2 credits*)
5. Show that the transformation $R \leftrightarrow 1/2R$, $k \leftrightarrow w$ is a symmetry of the spectrum. (*1 credit*)

Exercise 5.2: String on a Torus *13 Credits*

Next, we consider a sigma model of a string compactified on a two-torus. We absorb the radii in the metric such that the boundary conditions become

$$X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + 2\pi w^I\,, \qquad w^I \in \mathbb{Z}\,, \qquad I = 1, 2 \tag{5}$$

Then the action is given by

$$S = -\frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(G_{IJ}\eta^{\alpha\beta} - B_{IJ}\epsilon^{\alpha\beta} \right) \partial_\alpha X^I \partial_\beta X^J \tag{6}$$

where we assume constant metric $G_{IJ} = G_{JI}$ and Kalb–Ramond field $B_{IJ} = -B_{JI}$. Furthermore, $\eta = \mathrm{diag}(-1, 1)$ and $\epsilon^{01} = 1$.

1. Show that momentum quantization

$$\mathbb{Z} \ni k_I := \int_0^\infty p_I \mathrm{d}\sigma\,, \qquad \text{with} \quad p_I = \frac{\delta S}{\delta \dot{X}^I} \tag{7}$$

together with (5) imply that

$$p_L^I = w^I + G^{IJ}\left(\frac{1}{2}k_J - B_{JK}w^K\right)\,, \quad p_R^I = -w^I + G^{IJ}\left(\frac{1}{2}k_J - B_{JK}w^K\right)\,. \tag{8}$$

Hint: Expand $X^I_{L/R} = x^I_{L/R} + p^I_{L/R}\sigma_\pm$ *with neglecting oscillators.* $k_I = G_{IJ}\left(p_L^J + p_R^J\right) + B_{IJ}\left(p_L^J - p_R^J\right)$ *(3 credits)*

2. Show that the momentum contribution to the mass equation can be rewritten as

$$p_L^2 = G_{IJ}p_L^I p_L^J = \frac{1}{2T_2U_2}\left|(k_1 - Uk_2) - T(w^2 + Uw^1)\right|^2\,, \tag{9}$$

$$p_R^2 = G_{IJ}p_R^I p_R^J = \frac{1}{2T_2U_2}\left|(k_1 - Uk_2) - T^*(w^2 + Uw^1)\right|^2\,. \tag{10}$$

Here we defined the **Kähler–** and **complex structure moduli**

$$T = T_1 + \mathrm{i}T_2 := 2\left(B_{12} + \mathrm{i}\sqrt{G}\right)\,, \tag{11}$$

$$U = U_1 + \mathrm{i}U_2 := \frac{G_{12} + \mathrm{i}\sqrt{G}}{G_{22}}\,. \tag{12}$$

with $G = \det G_{IJ}$. *Hint: Deduce the* p_R^2 *result from* p_L^2*. Use as intermediate step*

$$\begin{aligned} p_L^2 = \frac{1}{G}\Big(&\frac{1}{4}(G_{22}k_1^2 + 2G_{12}k_1k_2 + G_{11}k_2^2) \\ &+ (B_{12}G_{12} - G)w_1k_1 - (B_{12}G_{12} + G)w_2k_2 + B_{12}G_{22}k_1w_2 - B_{12}G_{11}k_2w_1 \\ &+ (B_{12}^2 + G)(G_{11}w_1^2 + 2G_{12}w_1w_2 + G_{22}w_2^2)\Big)\,. \end{aligned}$$

(6 credits)

3. Show that the string spectrum is invariant under the following transformations:

- Modular Torus Transformation:

$$U \mapsto \frac{aU + b}{cU + d}\,, \qquad \text{generated by} \quad U \mapsto U + 1 \text{ and } U \mapsto -\frac{1}{U} \tag{13}$$

- T-Duality:

$$T \mapsto \frac{aT + b}{cT + d}\,, \qquad \text{generated by} \quad T \mapsto T + 1 \text{ and } T \mapsto -\frac{1}{T} \tag{14}$$

- Mirror Symmetry: $U \leftrightarrow T$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. How do the winding and momentum numbers have to transform? *(4 credits)*