Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 2
25 October 2013
WS 13/14

Exercises on Theoretical Particle Physics

Prof. Dr. H.-P. Nilles

–Home Exercises–
Due 1 November 2013

Important Remark

From now on and along the rest of the course we will employ the “mostly minus” prescription for the metric

$$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\,, \tag{1}$$

which is commonly used in quantum field theory.

H 2.1 The Dirac Equation *1.5+1+0.5+2+1+2+1.5=10 points*

Using the operator substitutions $\vec{p} \to -\mathrm{i}\vec{\nabla}$, $E \to \mathrm{i}\partial_t$ it is possible to get the equations for quantum mechanics from the energy-momentum relations. From the non-relativistic equation $E = \frac{p^2}{2m}$ one obtains the Schrödinger equation.

(a) Obtain the Klein-Gordon equation from the relativistic energy-momentum relation $E^2 = \vec{p}^2 + m^2$. Dirac's basic idea was to “factorize” the Klein-Gordon equation to obtain an equation which is first-order in the derivatives.

(b) Make the ansatz

$$H\psi = (\alpha_i p^i + m)\psi\,. \tag{2}$$

Squaring the Hamilton operator eq. (2) and using $H^2\psi = E^2\psi$ should give the Klein-Gordon equation. Show that from this requirement it follows:

$$\beta^2 = \alpha_i^2 = \mathbb{1} \quad \{\beta, \alpha_i\} = \{\alpha_i, \alpha_j\} = 0\,, \quad i \neq j \tag{3}$$

(c) Why are the α_i and the β not numbers? Why do they have to be hermitian ($A = A^\dagger$)? What does it imply?

(d) Define the Dirac gamma matrices γ^μ, $\mu = 0, ..., 3$ by

$$\gamma^0 = \beta\,, \quad \gamma^i = \beta\alpha_i \quad i = 1, 2, 3\,. \tag{4}$$

Show that the Dirac equation $H\psi = E\psi$ can be written in the covariant form

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0\,. \tag{5}$$

(e) Show that the gamma matrices fulfill the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\,, \tag{6}$$

with $\eta^{\mu\nu}$ as given in equation (1).

(f) Show the following relations:

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0\,, \quad (\gamma^k)^\dagger = -\gamma^k \\ (\gamma^0)^2 &= \mathbb{1}\,, \quad (\gamma^k)^2 = -\mathbb{1}\,, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \end{aligned} \tag{7}$$

The lowest dimensional matrices satisfying the Clifford algebra eq. (6) are 4×4 matrices. The choice of the matrices is not unique. The following are two possible representations: The Weyl (or chiral) representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{8}$$

and the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{9}$$

Here σ_1, σ_2 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

which satisfy the anti-commutation relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_{2\times2}\,. \tag{11}$$

(g) Verify that each set of matrices eqs. (8), (9) fulfills the Clifford algebra (6).

H 2.2 γ-Matrix identities *1.5+5+3.5=10 points*

The following exercise is to be solved by only using the Clifford algebra of the γ-matrices and **not** a particular representation. For convenience we introduce the notation

$$\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,.$$

(a) Show that

$$\left(\gamma^5\right)^\dagger = \gamma^5\,, \qquad \left(\gamma^5\right)^2 = \mathbb{1}\,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0\,.$$

(b) Prove the following trace theorems.

$$\begin{aligned} \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4\eta^{\mu\nu} \\ \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \\ \mathrm{tr}\left(\gamma^{\mu_1}\ldots\gamma^{\mu_n}\right) &= 0\,, \qquad \text{for } n \text{ odd} \\ \mathrm{tr}\,\gamma^5 &= 0 \\ \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \\ \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) &= -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

Hint: Use the cyclicity of the trace.

(c) Show the following contraction identities:

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 4 \cdot \mathbb{1} \\
\gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4\eta^{\nu\rho}\, \mathbb{1} \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu
\end{aligned}$$