

Exercises on Theoretical Particle Physics

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–HOME EXERCISES–
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H 8.1 The See–Saw mechanism

1+1+1+1+1+2+2+3=12 points

We write a four component Dirac spinor in the chiral representation as a composition of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

A *Majorana spinor* is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\bar{\Psi}^T = \Psi, \tag{1}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation operator.

- (a) Show that $(\Psi^c)^c = \Psi$.
- (b) What does eq. (1) imply for ψ_L and ψ_R and what is the physical meaning of this condition?
- (c) The Lagrangian \mathcal{L}_D for a Dirac spinor has the form

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu\partial_\mu)\Psi - m\bar{\Psi}\Psi,$$

where the second term is called the *Dirac mass term*. Rewrite \mathcal{L}_D in ψ_L and ψ_R .

- (d) Using the result of (b) rewrite the action \mathcal{L}_M for a Majorana spinor in terms of $\psi_{L/R}$

$$\mathcal{L}_M = \bar{\Psi}(i\gamma^\mu\partial_\mu)\Psi - \frac{m}{2}\bar{\Psi}\Psi.$$

The second term is called the *Majorana mass term*. Why is the factor 1/2 included in the mass term?

- (e) Remember the projectors $P_{L/R} = 1/2(1 \mp \gamma^5)$. As you know $P_{L/R}$ project Ψ onto the left/right handed part, respectively. We denote $\Psi_{L/R} := P_{L/R}\Psi$. Show that

$$\begin{aligned} (\Psi_{L/R})^c &= (\Psi^c)_{R/L} \\ \overline{(\Psi_{L/R})^c} &= \overline{\Psi_{R/L}} \end{aligned}$$

- (f) The most general mass term for a Dirac spinor is the *Dirac–Majorana mass term*,

$$\mathcal{L}_m = -\frac{1}{2} \left[2m_D \bar{\Psi}_L \Psi_R + m_L \bar{\Psi}_L (\Psi^c)_R + m_R \overline{(\Psi^c)_L} \Psi_R \right]$$

Show that this can be written in matrix form as

$$\mathcal{L}_m = -\frac{1}{2} \begin{pmatrix} \bar{\Psi}_L & \overline{(\Psi^c)_L} \end{pmatrix} \mathcal{M} \begin{pmatrix} (\Psi^c)_R \\ \Psi_R \end{pmatrix}$$

with

$$\mathcal{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$$

being the *neutrino mass matrix*.

- (g) Argue that in the SM extended by right-handed neutrinos, m_L must be zero and m_D is of the order of the electroweak symmetry breaking scale $M_W \sim 100$ GeV. We further assume that m_R is generated by some unspecified symmetry breaking mechanisms occurring at high energies, i. e. $m_R \sim M_{\text{GUT}} \sim 10^{16}$ GeV.
- (h) In this setup, diagonalize \mathcal{M} using an orthogonal matrix A

$$A^T \mathcal{M} A = \text{diag}(m_1, m_2).$$

Show that to the first non-vanishing order in the (small) parameter $\rho := m_D/m_R$ that the eigenvalues are $m_1 = -m_D^2/m_R$ and $m_2 = m_R$. Find the rotation matrix A to the first order in ρ for the diagonalization. What does $\rho \ll 1$ imply for the mass eigenstates? Insert the estimations done in (g) and compare the mass of the light neutrino to actual experimental bounds.

We see that by making one mass heavy the other one becomes very light. For this reason setups of this kind are generically referred to as *See-Saw mechanism*.

H 8.2 Neutrino oscillations

1+1.5+2+1.5+2=8 points

In analogy to the mixing of the quarks through weak interactions via the CKM matrix one can imagine a similar situation with the leptons once the neutrinos get mass. Hence, let us assume that there are n orthonormal flavor (interaction) eigenstates $|\nu_\alpha\rangle$. These states are transformed into n mass eigenstates ν_i via the unitary mixing matrix U ,

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle.$$

- (a) Assuming that the mass eigenstates $|\nu_i\rangle$ are stationary states and were emitted with momentum p by a source at $x = 0$ at $t = 0$, what is the form of $|\nu_i(x, t)\rangle$?
- (b) What is the relativistic Hamiltonian for a particle? For a highly relativistic particle we have $m \ll p$. Expand the Hamiltonian to first non-vanishing order in m/p .

- (c) A neutrino detector is built at a distance L from a source producing neutrinos in an eigenstate $|\nu_\alpha\rangle$. Show that the amplitude of detecting a neutrino in an eigenstate $|\nu_\beta\rangle$ is

$$A(\alpha \rightarrow \beta)(L) = \sum_i U_{\beta i}^* U_{\alpha i} \exp\left\{i \frac{m_i^2}{2} \frac{L}{E}\right\}.$$

Hint: For a highly relativistic particle you can set $v = 1 (= c)$ and $p \cong E$.

- (d) Obtain the transition probability P in terms of the differences of the mass squares $\Delta m_{ij}^2 := m_i^2 - m_j^2$. What is the probability of finding the original flavor?
- (e) Now assume that we have two flavors and one mixing angle θ . What is the form of U ? Compute $P(\alpha \rightarrow \beta)$ and $P(\alpha \rightarrow \alpha)$ for this case. Under which condition can one flavor completely rotate into another one?