Physikalisches Institut
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Theoretische Physik

Exercises 5
14.11.2016
WS 2016/17

Exercises on Theoretical Particle Physics I

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Due 21.11.2016

8. Yang-Mills theory (*7 credits*)

(a) Let us take a Lagrangian with a free Dirac fermion

$$\mathscr{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi$$

with Ψ transforming under a global SU(N) as

$$\Psi \to \Psi' = U\Psi \,, \quad U = \exp\left(i\,\alpha^a\, T^a\right) , \quad U^\dagger U = 1 \,.$$

T^a are the generators of SU(N) and α^a parametrises the gauge transformation. Show that $\mathscr{L}$ is invariant under this transformation.

(*1 credit*)

(b) As a next step we introduce local SU(N) transformations

$$\Psi \to \Psi' = U(x)\Psi \,, \quad U(x) = \exp\left(i\,\alpha^a(x)\, T^a\right) , \quad U^\dagger(x) U(x) = 1 \,.$$

Show that the transformation of $\mathscr{L}$ now leads to an extra term

$$\bar{\Psi}\, U^\dagger(x) i \gamma^\mu \left(\partial_\mu U(x)\right) \Psi$$

which means that $\mathscr{L}$ is not invariant under local SU(N) transformations.

(*1 credit*)

(c) The covariant derivative is defined via the requirement that $D_\mu \Psi$ transforms in the same way as Ψ itself

$$D_\mu \Psi = \left(\partial_\mu + i g A^a_\mu T^a\right) \Psi, \qquad D_\mu \Psi \to \left(D_\mu \Psi\right)' = U(x) D_\mu \Psi.$$

Show that this is equivalent to demanding that the gauge boson transforms as

$$A^a_\mu \to \left(A^a_\mu\right)' = A^a_\mu - f^{abc} \alpha^b A^c_\mu - \frac{1}{g} \partial_\mu \alpha^a$$

using a series expansion. How is f^{abc} defined?

(*1 credit*)

(d) Use part (c) to show that the following Lagrangian is gauge invariant

$$\mathscr{L} = \bar{\Psi} i \gamma^\mu D_\mu \Psi \,.$$

(*1 credit*)

(e) We define the field strength tensor F through

$$i\,g\left(F^a_{\mu\nu}T^a\right)\Psi = \left(D_\mu D_\nu - D_\nu D_\mu\right)\Psi.$$

Show that this definition leads to the following expression for its components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,f^{abc}A^b_\mu A^c_\nu.$$

(1 credit)

(f) Use part (e) to derive the transformation property of the field strength tensor

$$F_{\mu\nu} \to (F_{\mu\nu})' = U\,F_{\mu\nu}\,U^{-1}$$
$$F^a_{\mu\nu} \to \left(F^a_{\mu\nu}\right)' = F^a_{\mu\nu} + f^{abc}F^b_{\mu\nu}\alpha^c$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a$. Because of the last equation the field strength tensor itself is not gauge invariant.

(1 credit)

(g) Verify that the product

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

is gauge invariant and thus the gauge invariant Dirac Lagrangian is

$$\mathscr{L} = \bar{\Psi}i\gamma^\mu D_\mu\Psi - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right).$$

(1 credit)

9. The Standard Model Higgs effect *(13 credits)*

(a) The Glashow–Weinberg–Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. In a one-family approximation, the SM has the following particle content

	$L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	$R = e_R$	$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix}$	$T^aW^a_\mu$	B_μ
Hypercharge Y	-1	-2	$+1$	0	0
$\mathrm{SU}(2)_L$ rep.	**2**	**1**	**2**	**3**	**1**
Lorentz rep.	$(1/2, 0)$	$(0, 1/2)$	$(0, 0)$	$(1/2, 1/2)$	$(1/2, 1/2)$

where L, R contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{Higgs}} + \mathscr{L}_{\text{Yukawa}}$$

with

$$\mathscr{L}_{\text{gauge}} = \overline{R}(i\gamma^\mu D_\mu)R + \overline{L}(i\gamma^\mu D_\mu)L - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$
$$\mathscr{L}_{\text{Higgs}} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$
$$\mathscr{L}_{\text{Yukawa}} = -G_e\left(\overline{L}\Phi R + \overline{R}\Phi^\dagger L\right)$$

and

$$
\begin{aligned}
D_\mu &= \partial_\mu + ig'\frac{Y}{2}B_\mu + igT^a W^a_\mu \\
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu.
\end{aligned}
$$

Write down how the covariant derivative acts on the left- and right-handed lepton doublets/singlet and on the Higgs-doublet.

(*1 credit*)

(b) Show that the Lagrangian $\mathscr{L}$ given in part (a) is Lorentz invariant.

(*2 credits*)

(c) Show that $\mathscr{L}$ is also gauge invariant.

(*2 credits*)

(d) For the Higgs mechanism to work we need $\mu^2 < 0$. For which value of $|\Phi|$ does the Higgs potential obtain a minimum? In unitary gauge we can choose

$$
\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}
$$

where v is the vacuum expectation value and $\eta(x)$ a real field. Show that the Higgs potential around the minimum in the unitary gauge is given by

$$
V(\Phi) = -\mu^2\eta^2(x) + \lambda\, v\, \eta^3(x) + \frac{\lambda}{4}\eta^4(x)\,.
$$

What is the mass of the η field? Compare the degrees of freedom in the Higgs sector before and after spontaneous symmetry breaking.

(*2 credits*)

(e) Consider the kinetic energy terms of the Higgs field in $\mathscr{L}_{\text{Higgs}}$. Show that

$$
\begin{aligned}
(D_\mu\Phi)^\dagger(D^\mu\Phi) &= \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{4}g^2\,(v+\eta)^2\,W^-_\mu W^{+\,\mu} \\
&+ \frac{1}{8}\,(v+\eta)^2\begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\begin{pmatrix} g^2 & -\,g'g \\ -g'g & g'^{\,2} \end{pmatrix}\begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix}
\end{aligned}
$$

with $W^{\pm\,\mu} = \frac{1}{\sqrt{2}}(W^{1\,\mu} \mp i\,W^{2\,\mu})$.

(*2 credits*)

(f) What are the masses for $W_\mu^\pm$? To see the masses of W_μ^3 and B_μ bosons one has to diagonalize the matrix given in part (e)

$$\frac{v^2}{8}\begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix}\mathcal{O}^T\mathcal{O}\begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix}\mathcal{O}^T\mathcal{O}\begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix} = \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}.$$

Determine this orthogonal matrix $\mathcal{O}$ by computing the corresponding eigenvalues and eigenvectors. What are the masses of the Z_μ and A_μ fields? Compare the degrees of freedom in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of degrees of freedom?

(2 credits)

(g) As you know, an orthogonal 2×2 matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}.$$

Write $\cos\theta_W$ in terms of g' and g. Show for the ratio of the W- and Z-boson masses

$$\frac{m_W}{m_Z} = \cos\theta_W\,.$$

The angle θ_W is called Weinberg angle or weak mixing angle.

(1 credit)

(h) Finally, we want to consider the covariant derivative again. Substitute the fields B_μ and W_μ^a by $W_\mu^\pm$, Z_μ and A_μ and show

$$D_\mu = \partial_\mu + i\,A_\mu\,e\,Q + iZ_\mu\frac{1}{\sqrt{g^2+g'^2}}\left(g^2T_3 - g'^2\frac{Y}{2}\right) + \frac{ig}{\sqrt{2}}\begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix}$$

where we have defined the electric charge $e = \frac{g'g}{\sqrt{g^2+g'^2}}$ and $Q = T_3 + \frac{Y}{2}$.

(1 credit)