Advanced Quantum Theory (WS 24/25)
Homework no. 10 (December 9, 2024)
Please hand in your solution by Sunday, December 15!

Quickies

Q1: *(i)* Does the Born approximation for the scattering cross section depend on the sign of the scattering potential (attractive or repulsive)? *(ii)* Does the existence of bound states depend on this sign? *(iii)* What does this imply for the ability of the Born approximation to predict resonances in the scattering cross section? **[3P]**

Q2: At small momenta, the total cross section for the ℓth partial wave is $\propto |\vec{k}|^{4\ell}$. What does this imply for the scaling of *(i)* the partial wave amplitude a_ℓ; *(ii)* the scaling of the scattering phase δ_ℓ with $|\vec{k}|$ at small $|\vec{k}|$? **[2P]**

Q3: Can the scattering formalism developed in class so far (scattering of a non–relativistic particle on a fixed scattering center) be applied to the following reactions, and, if so, for what range of energy of the incoming particle: *(i)* electron–proton scattering; *(ii)* neutron scattering on a lead nucleus; *(iii)* proton–proton scattering? **[3P]**

1) Potential Well and Born Approximation

In this exercise we analyze scattering on a spherically symmetric potential well, as in sec. 5.5 in class, i.e. $V(r) = -V_0$ for $r \leq r_0$ while $V(r) = 0$ for $r > r_0$. M is the mass of the scattering particle.

1. Show that in Born approximation the scattering amplitude $f_{\vec{k}}(\theta, \phi)$ can be written as

$$f_{\vec{k}}^{\rm Born} = \frac{2MV_0 r_0}{\hbar^2 |\vec{q}|^2} \left[\frac{1}{r_0|\vec{q}|} \sin(r_0|\vec{q}|) - \cos(r_0|\vec{q}|) \right] , \tag{1}$$

where $\vec{q} = \vec{k}' - \vec{k}$, i.e. $\hbar\vec{q}$ is the momentum exchanged in the scattering. Hint: Recall eq.(5.22) in class,

$$f_{\vec{k}}^{\rm Born} = -\frac{2M}{\hbar^2|\vec{q}|} \int_0^\infty rV(r) \sin(|\vec{q}|r) dr\,. \tag{2}$$

[3P]

2. Show that for $|\vec{q}|r_0 \ll 1$,

$$f_{\vec{k}}^{\rm Born} \simeq v_0 \frac{r_0}{3}\,, \tag{3}$$

where

$$v_0 = \frac{2MV_0 r_0^2}{\hbar^2}\,, \tag{4}$$

see eq.(5.58) in class. *Hint:* You need to expand the trigonometric functions in eq.(1) up to the second term. **[3P]**

3. Show that therefore in the same limit, the total scattering cross section is

$$\sigma^{\rm Born} \simeq 4\pi r_0^2 \frac{v_0^2}{9}\,. \tag{5}$$

Interpret this result. In particular, show that in this limit only the $S-$wave contributes, and that $|\vec{q}|r_0 \ll 1$ requires $|\vec{k}|r_0 \ll 1$. **[3P]**

4. Recall that the Born approximation means that one uses the incoming (plane) wave when computing the contribution of the spherical wave to the entire wave function; from eq(5.12), this contribution is given in Born approximation by

$$\psi_{\rm scat}^{\rm Born}(\vec{x}) = -\frac{2M}{\hbar}\int d^3x' \frac{{\rm e}^{i|\vec{k}||\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}V(\vec{x}'){\rm e}^{i\vec{k}\cdot\vec{x}'}\,. \tag{6}$$

Argue that the Born approximation can only be trusted if $|\psi_{\rm scat}^{\rm Born}(\vec{x}=0)| \ll 1$. Apply this to our potential well in the limit $|\vec{k}|r_0 \ll 1$, and show that it is equivalent to

$$\frac{v_0}{2} \ll 1\,, \tag{7}$$

where v_0 has been defined in eq.(4). **[5P]**

5. For the case at hand the scattering cross section can be computed exactly. In particular, from eq.(5.69) in class we have for the $S-$wave scattering phase:

$$\delta_0 = \arctan\left[\frac{k}{q}\tan(qr_0)\right] - kr_0\,, \tag{8}$$

where

$$qr_0 = \sqrt{(kr_0)^2 + v_0}\,, \tag{9}$$

see eq.(5.70) in class, i.e. $\hbar q$ is *not* the absolute value of the momentum exchange. Show that eq.(8) reproduces eq.(5) in the appropriate limit $kr_0,\ v_0 \ll 1$. *Hint:* You again need to keep the two leading terms in the expansion of the trigonometric functions in eq.(8). **[5P]**