Advanced Quantum Theory (WS 24/25)
Homework no. 13 (January 13, 2025)
Please send in your solution by Monday, January 20!

Quickies

Q1: What (anti)commutation relations do the operators creating or annihilating a boson or fermion in a given state satisfy? *Hint:* Assume a discrete spectrum of states, i.e. use Kronecker$-\delta$ rather than Dirac$-\delta$. **[3P]**

Q2 Evaluate $\hat{a}_i^\dagger |n_1, n_2, \dots\rangle$ and $\hat{b}_i^\dagger |n_1, n_2, \dots\rangle$; here $\hat{a}_i^\dagger$ creates a boson in state i, $\hat{b}_i^\dagger$ creates a fermion in state i, and $|n_1, n_2, \dots\rangle$ has n_1 particles in state 1, n_2 particles in state 2 and so on. **[2P]**

Q3 What is the *total* particle number operator in second quantization? **[1P]**

1 Lorentz Transformations and the Klein–Gordon Equation

The Klein–Gordon equation can be written as

$$\partial_\mu \partial^\mu \phi(x) = -\frac{m^2 c^2}{\hbar^2} \phi(x) \,, \tag{1}$$

where $\partial_\mu = \partial / \partial x^\mu$ and $\partial^\mu = \partial / \partial x_\mu$.

1. Now consider a different inertial frame, described by coordinates $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Show that
$$\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'_\mu} \frac{\partial}{\partial x'^\mu} \,. \tag{2}$$
Hint: Use the chain rule, and the relation defining a Lorentz transformation, $(\Lambda^T g \Lambda)_{\mu\nu} = g_{\mu\nu}$, where $g_{\mu\nu}$ is the Minkowski metric. **[2P]**
2. Now consider a boost in x direction. Show by explicit calculation that $\partial^2/\partial(ct)^2 - \partial^2/\partial x^2$ is Lorentz invariant, while $\partial^2/\partial(ct)^2 + \partial^2/\partial x^2$ is not. **[2P]**
3. Show that the plane wave, $\mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$, is Lorentz invariant. **[1P]**
4. For a general solution of the free Klein–Gordon equation we have to consider wave packets. Show that the integration measure d^3p is *not* Lorentz invariant, while $d^3p/(2E)$ is; here $\vec{p}$ is the 3$-$momentum and $E = \sqrt{\vec{p}^2 + m^2}$ is the energy. **[3P]**

2 Properties of the Pauli Matrices

The Pauli matrices are given by (I expect you to remember these!):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \tag{3}$$

where i is the imaginary unit ($i^2 = -1$). Prove the following relations:

1.

$$\sigma_k^2 = \mathbb{1} \; \forall k \in \{1,2,3\}\,, \tag{4}$$

where $\mathbb{1}$ is the 2×2 unit matrix. **[2P]**

2. $\sigma_k \sigma_l = -\sigma_l \sigma_k \; \forall l \neq k$, i.e. $\{\sigma_k, \sigma_l\} = 0 \; \forall l \neq k$; here $\{A, B\}$ is the anticommutator of matrices A and B. Show that this implies

$$[\sigma_l, \sigma_k] = 2\sigma_l \sigma_k \; \forall l \neq k\,, \tag{5}$$

where $[A, B]$ is the commutator of A and B. **[3P]**

3.

$$[\sigma_l, \sigma_k] = 2i \sum_{j=1}^{3} \epsilon_{lkj} \sigma_j \; \forall l, k \in \{1,2,3\}\,, \tag{6}$$

where ϵ_{ijk} is the totally antisymmetric rank–3 tensor, with $\epsilon_{123} = 1$. How many different j actually contribute to the sum? *Hint:* Consider $l = k$ and $l \neq k$ separately, and use the appropriate results from above. **[3P]**

4.

$$\sigma_k \sigma_l = \delta_{kl} \mathbb{1} + i \sum_{j=1}^{3} \epsilon_{klj} \sigma_j\,. \tag{7}$$

Hint: Again treat the cases $k = l$ and $k \neq l$ separately, and use the appropriate results from above. **[3P]**

5.

$$(\vec{\sigma} \cdot \vec{a})\,(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b}\,\mathbb{1} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})\,. \tag{8}$$

Here $\vec{a}$ and $\vec{b}$ are two arbitrary 3−vectors (in Euclidean space), and $\vec{\sigma} \cdot \vec{a} = \sum_{i=1}^{3} a_i \sigma_i$. *Hint:* Recall that the vector product satisfies $\left(\vec{a} \times \vec{b}\right)_i = \sum_{j,k} \epsilon_{ijk} a_j b_k$, and use the appropriate result from above. **[4P]**