Advanced Quantum Theory (WS 24/25)

Homework no. 4 (October 28, 2024)

To be handed in by Sunday, November 3, 2024

1 Free Particle Propagator 1: Short–Time Evolution

In class we saw that the propagator for a free particle in one spatial dimension is given by:

$$U(x, x', t) = \sqrt{\frac{m}{2i\pi\hbar t}} \exp\left[\frac{i(x - x')^2 m}{2\hbar t}\right] . \tag{1}$$

Here m is the mass of the particle, and the initial condition is set at $t_0 = 0$. This allows to compute the wave function at later times via

$$\psi(x, t) = \int_{-\infty}^{\infty} dx' U(x, x', t)\psi(x', 0) . \tag{2}$$

You may have noticed that U appears to be singular as $t \to 0$. Using eq.(2), argue that for $t \to 0$ only values of x' very near x are relevant. Expand $\psi(x', 0)$ around $x' = x$ up to second order, and show by explicit calculation that in the limit $t \to 0$, eq.(2) indeed reduces to $\psi(x, 0)$, i.e. the propagator reduces to a delta "function", $U(x, x', t) \to \delta(x - x')$. *Hint:* You will need a couple of Gaussian integrals from the previous HW sheet! **[4P]**

2 Free Particle Propagator 2: Late–Time Evolution

Now we wish to look at the evolution of a free field at late times. We assume that the wave function is initially a Gaussian.

1. The initial wave function has the form (again setting $t_0 = 0$)

$$\psi(x, 0) = N \exp\left[-\frac{(x - x_0)^2}{4\sigma^2}\right] . \tag{3}$$

   What is the value of N for a properly normalized wave function? What is the physical meaning of σ? **[2P]**

2. Now consider the late–time solution, i.e. eq.(2) for $\hbar t/m \gg \sigma^2$. You should find that $|\psi(x, t)|^2$ again has Gaussian form, where the width scales $\propto t/(m\sigma)$. Explain this scaling! *Hint:* Consider the classical motion of a free particle, and remember the uncertainty relation! **[4P]**

3 Bonus Problem

Ask your tutor at least one question (with non–obvious answer) to one of the previous lectures!