Advanced Quantum Theory (WS 25/25)

Homework no. 6 (November 11, 2024)

To be handed in by Sunday, November 17

Quickies

Q1: How does the propagator $U(x, x', t, t_0)$ relate the wave function $\psi(x, t)$ to the (known) wave function $\psi(x, t_0)$? **[1P]**

Q2: Write the propagator $U(x, x', t, t_0)$ in terms of a coordinate path integral. *Hint:* You need not specify the pre–factor. **[1P]**

Q3: *(i)* What properties does the propagator operator $\hat{U}(t, t_0)$ have? *(ii)* How is the function (integration kernel) $U(x, x', t, t_0)$ appearing in the first two questions related to the operator $\hat{U}(t, t_0)$? **[2P]**

1) Exponentiating Operators

The exponential of an operator $\hat{A}$ is defined via the series expansion

$$\mathrm{e}^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\left(\hat{A}\right)^n}{n!} . \tag{1}$$

1. Show that $\mathrm{e}^{\hat{A}}\mathrm{e}^{-\hat{A}} = 1$, directly from the definition (1). **[3P]**
2. Consider a second operator $\hat{B}$ acting on the same Hilbert space. Assume that the commutator $[\hat{A}, \hat{B}] = c$ is a complex number. In this case the Baker–Campbell–Hausdorff formula reduces to $\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}\mathrm{e}^{-[\hat{A},\hat{B}]/2}$. Show that in this case, $\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}} = \mathrm{e}^{\hat{B}}\mathrm{e}^{\hat{A}}\mathrm{e}^{[\hat{A},\hat{B}]}$. **[2P]**
3. Show that for general $\hat{A}$, $\hat{B}$ the following identity holds:

$$\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, \hat{B}]_n , \tag{2}$$

where the generalized commutator is defined by

$$[\hat{A}, \hat{B}]_n = [\hat{A}, [\hat{A}, \hat{B}]_{n-1}] = \hat{A}[\hat{A}, \hat{B}]_{n-1} - [\hat{A}, \hat{B}]_{n-1}\hat{A} \tag{3}$$

for $n > 0$, with $[\hat{A}, \hat{B}]_0 = \hat{B}$. *Hint:* Show by induction that

$$[\hat{A}, \hat{B}]_n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \hat{A}^{n-i}\hat{B}(-\hat{A})^i ,$$

and compare with the expansion of the left–hand side of eq.(2). **[4P]**

2) First Order Time Dependent Perturbation Theory

In class we had derived the first–order expression for the transition probability from the general n–th order result. Here we want to re–derive this expression in a more direct manner.

Starting point is the expansion of the wave function in terms of eigenstates $|n^{(0)}\rangle$ of the *unperturbed, time–independent* Hamiltonian $\hat{H}_0$:

$$|\psi(t)\rangle = \sum_n c_n(t)|n^{(0)}\rangle . \tag{4}$$

We want to compute the probability P_{fi} for an $|i^{(0)}\rangle \to |f^{(0)}\rangle$ transition, i.e. the probability that a state $|\psi(t_0)\rangle = |i^{(0)}\rangle$ at initial time t_0 transits to state $|f^{(0)}\rangle$ at time t. Since the $|n^{(0)}\rangle$ form a complete basis, an expansion as in (4) is always possible.

1. Argue that $P_{fi}(t) = |c_f(t)|^2$, where the $c_n(t)$ satisfy the boundary condition $c_n(t_0) = \delta_{in}$. **[1P]**

2. It is convenient to rewrite eq.(4) as
$$|\psi(t)\rangle = \sum_n d_n(t)|\psi_n^{(0)}(t)\rangle\,, \tag{5}$$
where $|\psi_n^{(0)}(t)\rangle$ is the time–dependent state as evolved using the *unperturbed* Hamiltonian $\hat{H}_0$, with initial condition $|\psi_n^{(0)}(t_0)\rangle = |n^{(0)}\rangle$. What is the relation between $d_n(t)$ and $c_n(t)$? **[2P]**

3. By using the Schrödinger equation for $|\psi(t)\rangle$, show that
$$i\hbar\dot{d}_f = \sum_n \langle f^{(0)}|\hat{H}_1(t)|n^{(0)}\rangle \mathrm{e}^{i\omega_{fn}(t-t_0)} d_n(t)\,, \tag{6}$$
where $\omega_{fn} = (E_f^{(0)} - E_n^{(0)})/\hbar$, with $E_k^{(0)}$ being the $k-$th eigenvalue of the unperturbed Hamiltonian $\hat{H}_0$. **[3P]**

4. Equation (6) is exact, but not yet very useful. Reproduce the first–order expression for $P_{fi}(t)$ by inserting the *zeroth–order* solution of (6) into the right–hand side of (6). **[4P]**

3) Perturbed Harmonic Oscillator

Consider a one–dimensional harmonic oscillator with characteristic frequency ω_0, which is perturbed by a small time–dependent perturbation.

1. First consider a perturbation
$$\hat{H}_1(t) = a\hat{x}^p \mathrm{e}^{-t^2/\tau^2}\,, \tag{7}$$
where a is a real constant (of appropriate unit), p is an integer, and τ characterizes the time during which the perturbation is active (since $\hat{H}_1(t)$ goes to zero both for $t \ll -\tau$ and for $t \gg \tau$). Assume that the system is in the ground state for $t \to -\infty$. Show that to first order in perturbation theory the perturbation (7) does not populate states $|f^{(0)}\rangle$ with $f > p$. **[2P]**

2. Use parity arguments to further reduce the number of states that can be populated by the perturbation (7) in first order perturbation theory. What states are accessible for even (odd) p? **[3P]**

3. Explicitly compute $P_{1,0}$ for transitions from the ground state at $t \to -\infty$ to the first excited state at $t \to +\infty$, for $p = 1$. What happens for $\tau \to 0$ if *(i)* a remains constant, *(ii)* a is varied proportional to $1/\sqrt{\tau}$? **[5P]**